1.1 - What is Physics?

Monday, March 13, 2017 9:04 AM

Matter

- Matter - anything that has mass and takes up space
- Mass - the amount of "stuff" making up an object
  - Stars
  - Electrons
  - Neil Diamond
- Inertial Mass
  - How hard it is to accelerate an object
- Gravitational Mass
  - How large a gravitational force an object experiences

Energy

- Energy - The ability or capacity to do work
- Work - the process of moving an object

Mass-Energy Equivalence

- In the early 20th century, Albert Einstein formalized a relationship between mass and energy
- The mass of an object, a key characteristic of matter, is really a measure of its energy
- The source of all energy on Earth is the conversion of mass into energy

1.2 - Math Review

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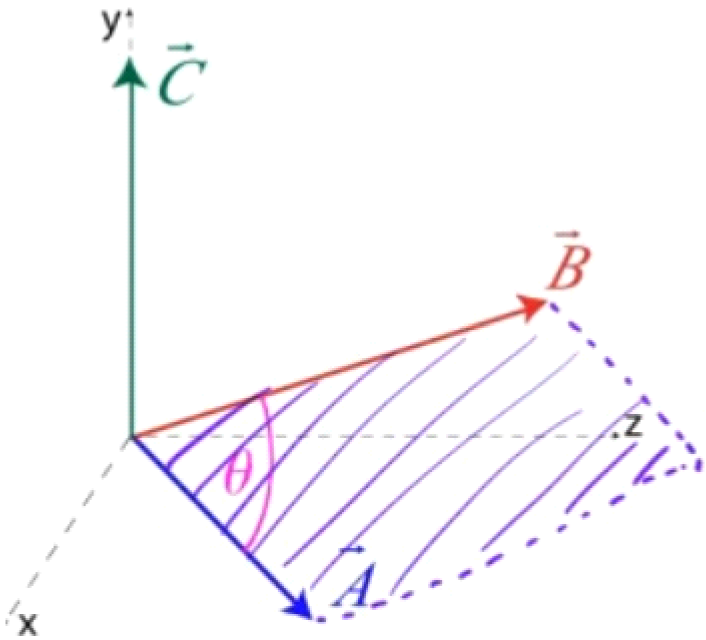
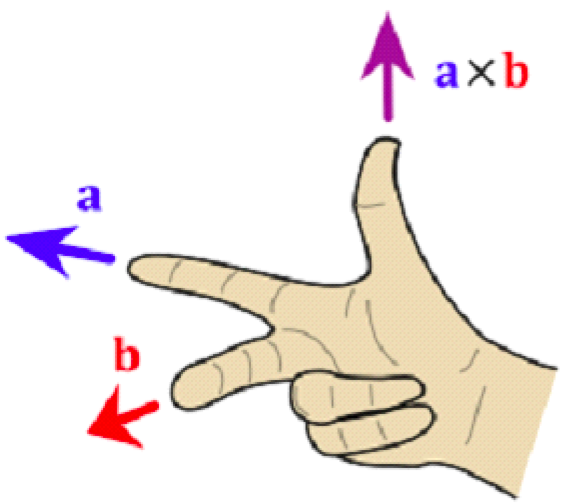
Cross Product

- Cross (vector) product of two vectors gives you a vector perpendicular to both whose magnitude is equal to the area of a parallelogram defined by the two initial vectors



- Positive direction of the cross product is given by the right-hand rule



- Cross product of parallel vectors is zero.

Calculating the Cross Product

- $\left|\vec{A} \times \vec{B}\right| = AB\sin\theta$
- $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

  $= \left(A_y B_z - A_z B_y\right)\hat{\imath} + \left(A_z B_x - A_x B_z\right)\hat{\jmath} + \left(A_x B_y - A_y B_x\right)\hat{k}$

Cross Product Properties

- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- $\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
- $c\left(\vec{A} \times \vec{B}\right) = \left(c\vec{A}\right) \times \vec{B} = \vec{A} \times \left(c\vec{B}\right)$
- $\frac{d}{dt}\left(\vec{A} \times \vec{B}\right) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$

Unites

S.No.	Fundamental Quantities	Fundamental Units	Symbol
1.	Length	metre	m
2.	Mass	kilogram	kg
3.	Time	second	S
4.	Temperature	kelvin	kg
5	Electric current	ampere	A
6	Luminous intensity	candela	cd
7	Amount of substance	mole	mol

2.1 - Describing Motion I

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Position / Displacement

- An object's position is its location at a given point in time
- The vector from the origin of the coordinate system to the object's position is known as the position vector, $\vec{r}$
- As an object moves, its position changes. This change in position is called displacement, $\Delta\vec{r}$
- Position and displacement are both vectors
- In one dimension, position is given by the x-coordinate, and displacement by $\Delta\vec{x}$

Average Speed

- Average speed is the distance traveled divided by the time interval
- $\bar{v} = \frac{x}{t}$
- Average speed is a scalar, and is measured in meters/second

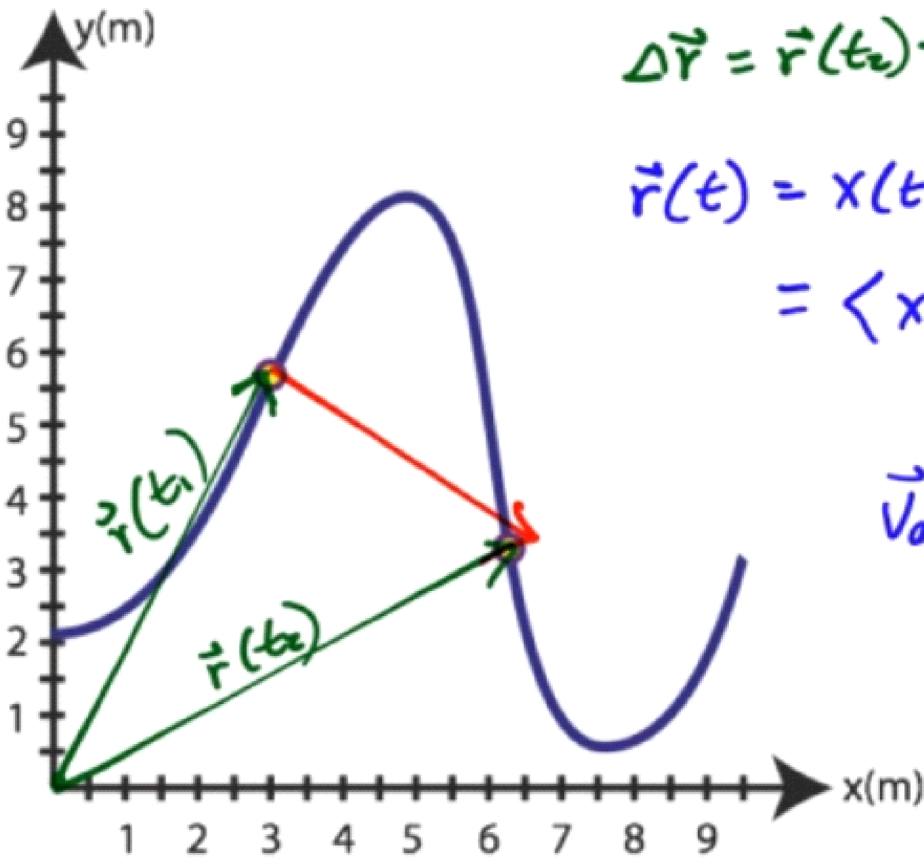
Average Velocity

- Velocity is the rate at which position changes.
- Average velocity is the displacement during a time interval divided by the time interval
- $v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$
- Average velocity is a vector, and is also measured in meter/second

Acceleration

- Acceleration is the rate at which velocity changes
- $a = \frac{\Delta v}{\Delta t}$
- Acceleration is a vector
- Units are m/s^2

The Position Vector

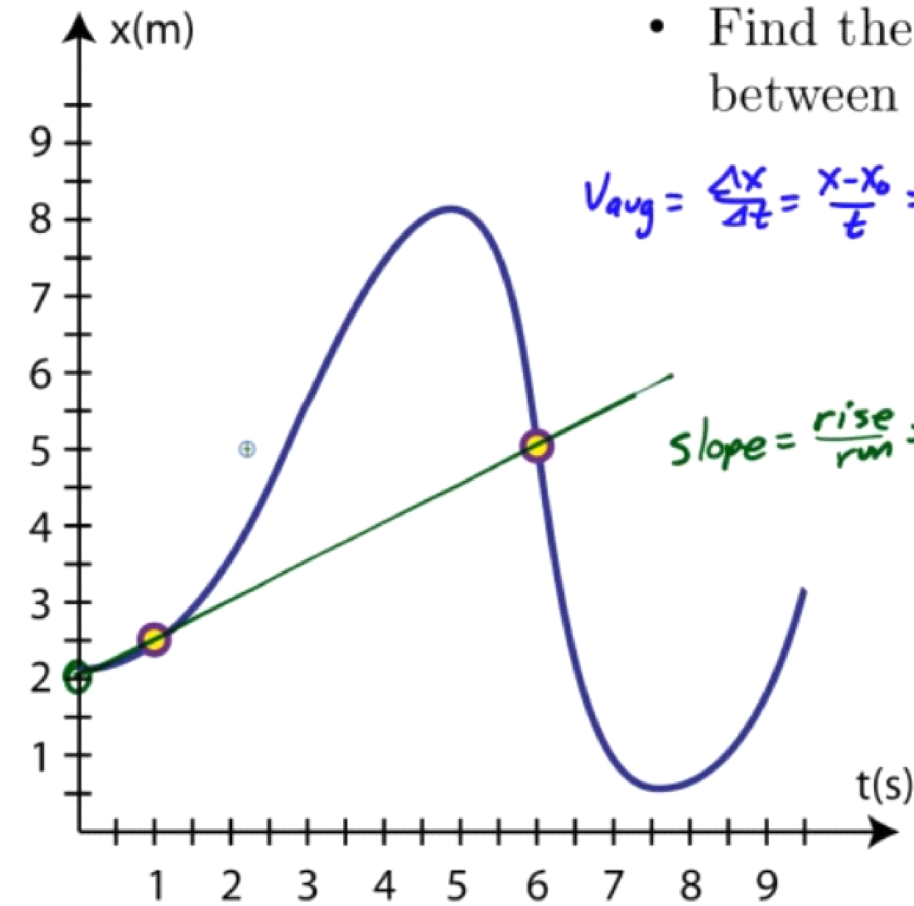


$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$$

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$
$$= \langle x(t), y(t) \rangle$$

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$$

Average Velocity



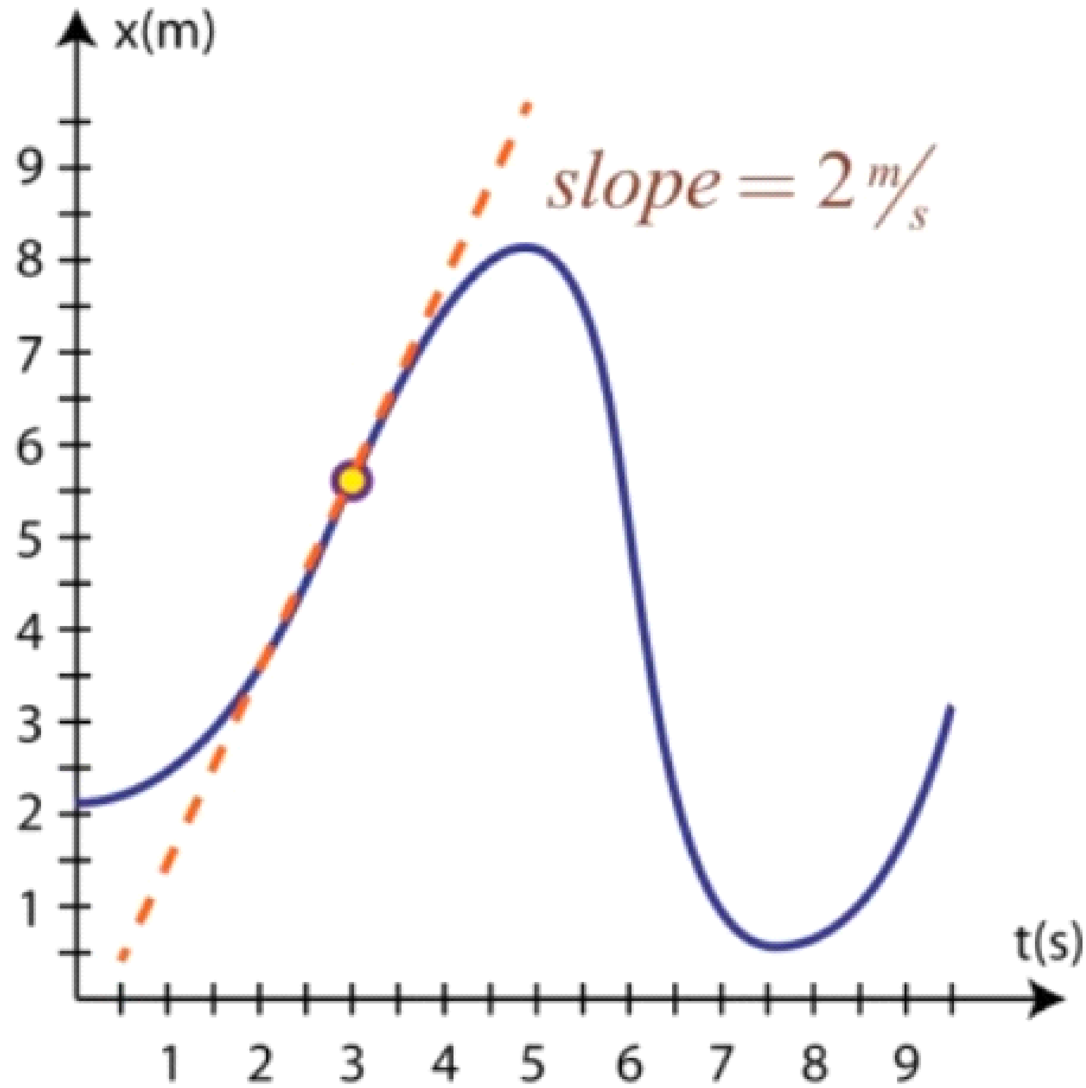
- Find the average velocity between 1 and 6 seconds.

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t} = \frac{5m - 2.5m}{6s - 1s} = 0.5\frac{m}{s}$$

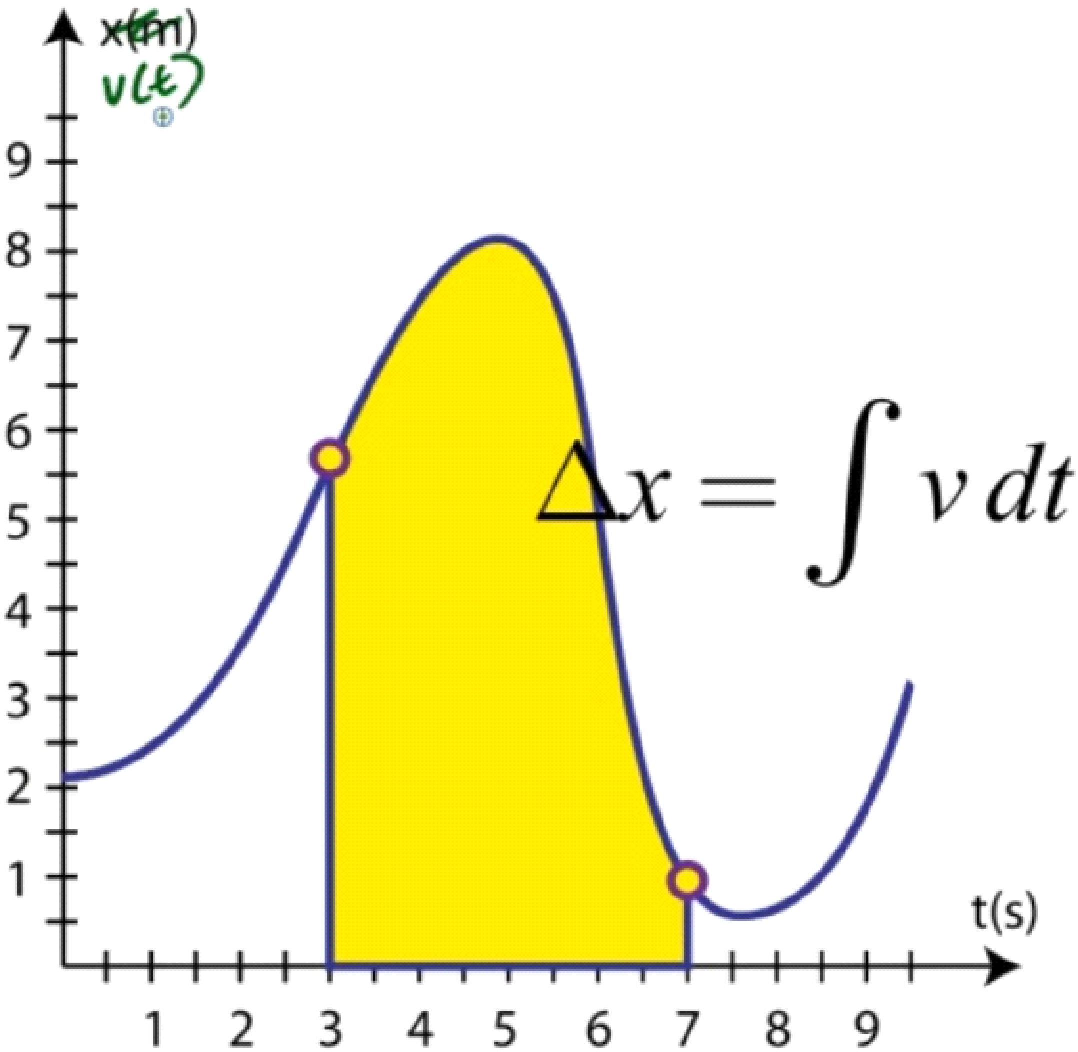
$$slope = \frac{rise}{run} = \frac{5m - 2m}{6s - 0s} = \frac{3m}{6s} = 0.5\frac{m}{s}$$

Instantaneous Velocity

- Average velocity observed over an infinitely small time interval provides instantons velocity
- Instantaneous velocity is the derivative of position with respect to time



- Area under the velocity-time graph is the displacement during that time interval

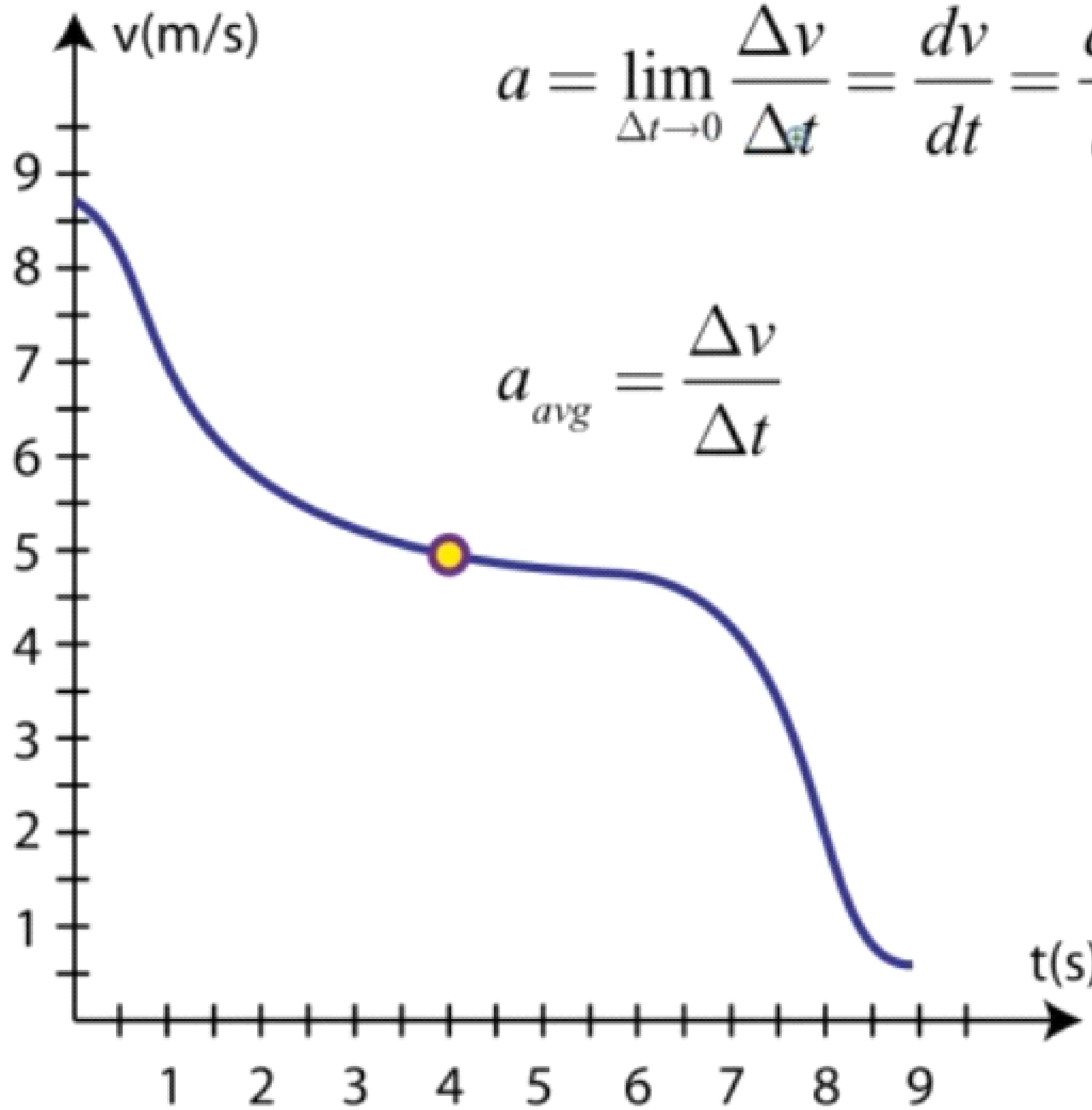


Acceleration

- Acceleration is the rate at which velocity changes

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$$

$$a_{avg} = \frac{\Delta v}{\Delta t}$$



2.2 - Describing Motion II

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Special Case: Constant Acceleration

- For cases of constant acceleration, we can derive a set of "kinematic equations" that will allow us to solve for unknown quantities

$$v = v_0 + at \qquad \text{no } \Delta x$$

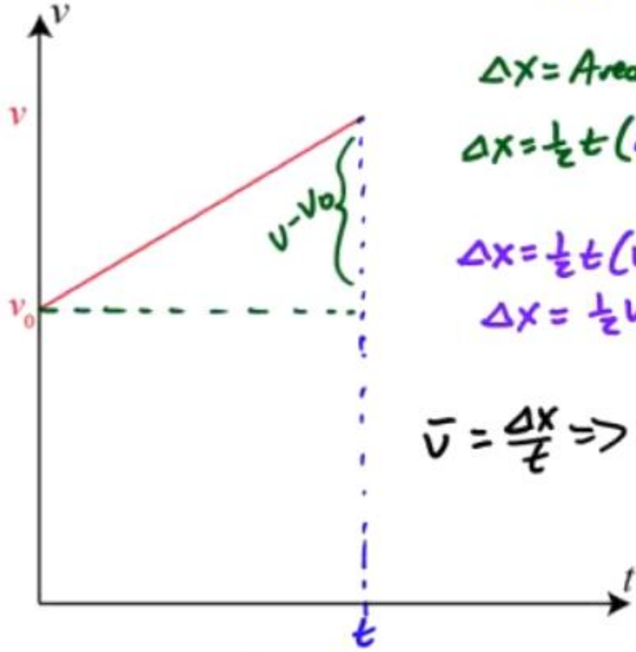
$$\Delta x = v_0 t + \tfrac{1}{2} a t^2 \qquad \text{no } v$$

$$v^2 = v_0^2 + 2a\Delta x \qquad \text{no } t$$

$$\Delta x = \bar{v} t = \tfrac{1}{2}\left(v + v_0\right) t \qquad \text{no } a$$

$$\Delta x = vt - \tfrac{1}{2} a t^2 \qquad \text{no } v_0$$

Deriving the Kinematic Equations



$$a = \frac{\Delta v}{t} = \frac{v-v_0}{t} \Rightarrow \boxed{v = v_0 + at} \quad ①$$

$$\Delta x = \text{Area} = \tfrac{1}{2}bh + lw = \tfrac{1}{2}t(v-v_0) + v_0 t \xrightarrow[v-v_0=at]{v=v_0+at}$$

$$\Delta x = \tfrac{1}{2}t(at) + v_0 t \Rightarrow \boxed{\Delta x = v_0 t + \tfrac{1}{2}at^2} \quad ②$$

$$\Delta x = \tfrac{1}{2}t(v-v_0) + v_0 t = \tfrac{1}{2}vt - \tfrac{1}{2}v_0 t + v_0 t \Rightarrow$$

$$\Delta x = \tfrac{1}{2}vt + \tfrac{1}{2}v_0 t \Rightarrow \frac{\Delta x}{t} = \frac{v+v_0}{2} \Rightarrow \boxed{\bar{v} = \frac{v+v_0}{2}}$$

$$\bar{v} = \frac{\Delta x}{t} \Rightarrow \Delta x = \bar{v}t = \left(\frac{v+v_0}{2}\right)t \xrightarrow[t=\frac{v-v_0}{a}]{v=v_0+at}$$

$$\Delta x = \left(\frac{v+v_0}{2}\right)\left(\frac{v-v_0}{a}\right) \Rightarrow$$

$$\Delta x = \frac{v^2-v_0^2}{2a} \Rightarrow v^2 - v_0^2 = 2a\Delta x \Rightarrow \boxed{v^2 = v_0^2 + 2a\Delta x} \quad ③$$

Free Fall

- When the only force acting on an object is the force of gravity, we refer to object's motion as free fall
- This includes objects that have a non-zero initial velocity

Air Resistance

- If we drop a ball and a sheet of paper, it is obvious they don't fall at the same rate
- If we could remove all the air from the room, however, we would find that they fall at the same rate
- We will analyze the motion of objects by neglecting air resistance (a form of friction) for the time being

Acceleration Due to Gravity

- Near the surface of Earth, objects accelerate downward at a rate of $9.8 m/s^2$
- We call this acceleration the acceleration due to gravity (g)
- More accurately, g is referred to as the gravitational field strength
- As you move further away from Earth, g decreases

2.3 - Projectile Motion

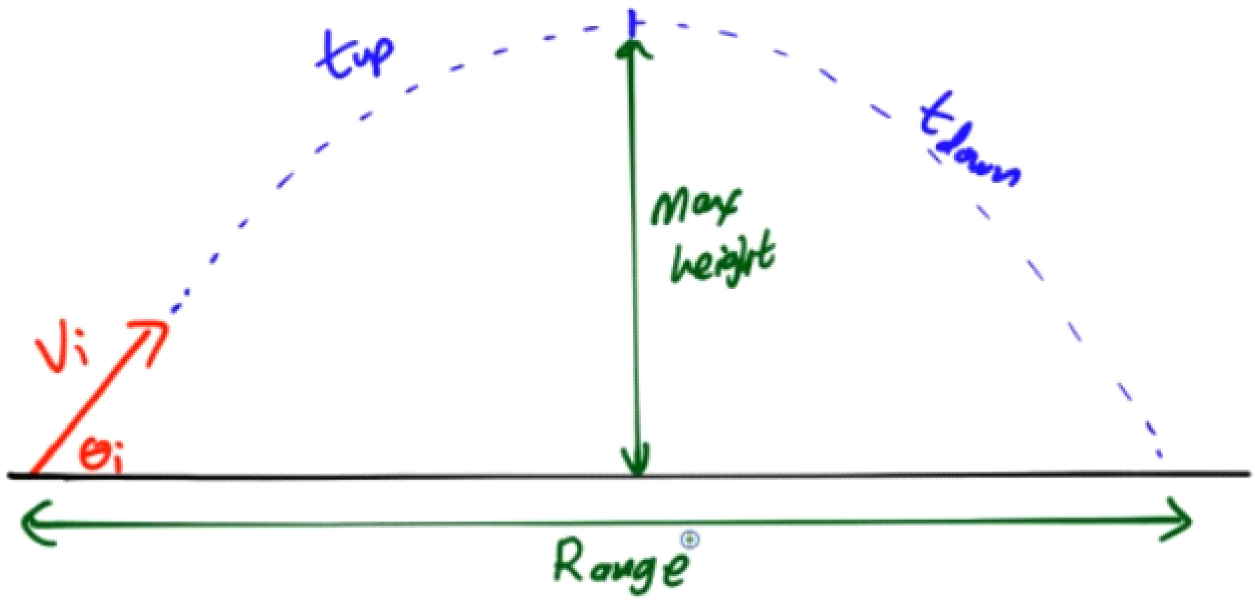
Monday, March 13, 2017 9:50 AM

What Is a Projectile

- A projectile is an object that is in acted upon only by gravity
- In reality, air resistance plays a role
- Typically, projectiles are objects launched at an angle

Path of a Projectile

- Projectiles launched at an angle move in parabolic arcs



Independence of Motion

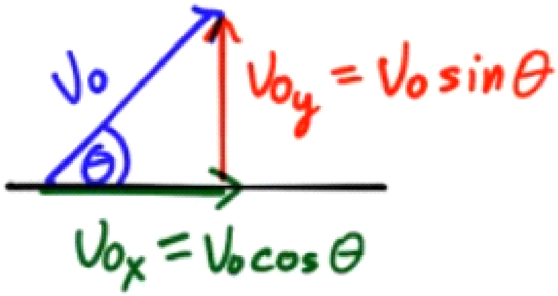
- Projectiles launched at an angle have motion in two dimensions
  - Vertical - like free fall
  - Horizontal - 0 acceleration
- Vertical motion and horizontal motion can be treated separately!

Angled Projectiles

- For objects launched at an angle, you must first break up the object's initial velocity into x and y components of initial velocity
- Then, use these components of initial velocity in your horizontal and vertical

motion tables

- An object will travel the maximum horizontal distance with a launch angle of 45°



Graphing Projectile Motion

- Track vector components as a function of time
- Solve for time in the horizontal, and use that to eliminate time in the vertical equation

Horz

$$\Delta x = V_{x_0} t \Rightarrow t = \frac{\Delta x}{V_{x_0}}$$

Vert

$$\Delta y = V_{y_0} t + \frac{1}{2} a_y t^2 \xRightarrow{t = \frac{\Delta x}{V_{x_0}}}$$

$$\Delta y = V_{y_0} \frac{\Delta x}{V_{x_0}} + \frac{1}{2} a_y \frac{\Delta x^2}{V_{x_0}^2} \Longrightarrow$$

$$y = \left(\frac{V_{y_0}}{V_{x_0}}\right) x + \left(\frac{a_y}{2V_{x_0}^2}\right) x^2$$

Parabola

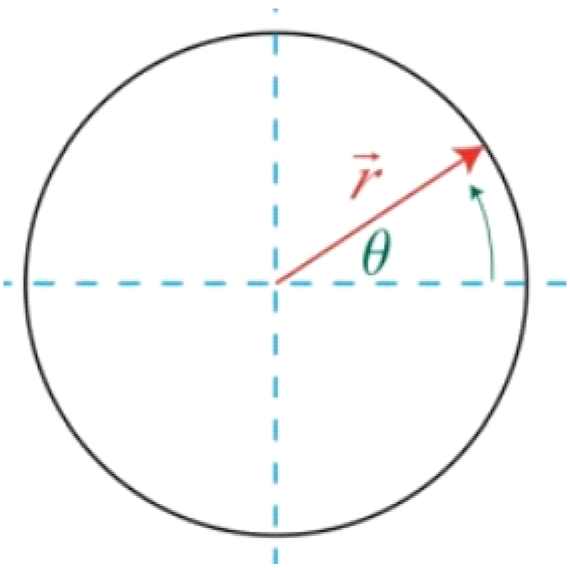
2.4 - Circular & Relative Motion

Monday, March 13, 2017 9:58 AM

Radians and Degrees

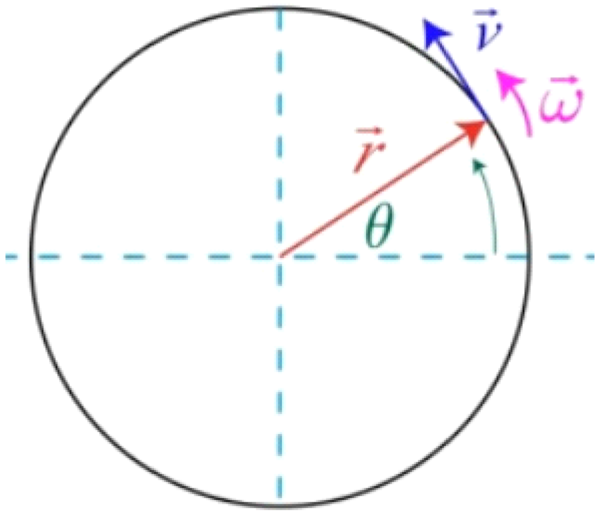
- In degrees, once around a circle is 360°
- In radians, once around a circle is 2π
- A radian measures a distance around an arc equal to the length of the arc's radius
- $\Delta s = C = 2\pi r$

Linear vs. Angular Displacement



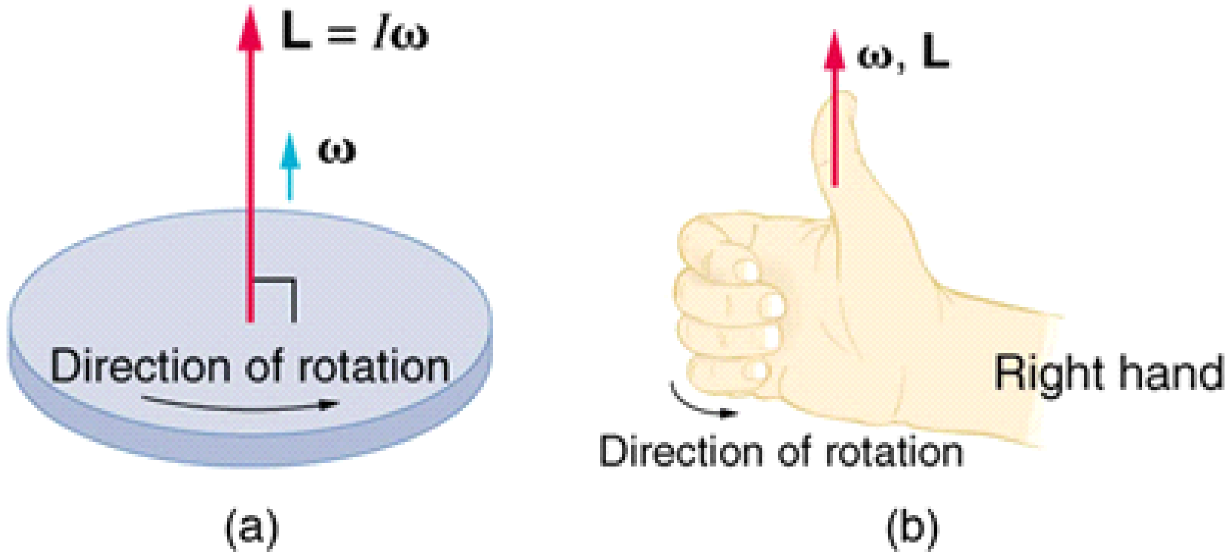
- Linear position / displacement given by Δr or Δs
- Angular position / displacement given by $\Delta\theta$
- $s = r\theta$
- $\Delta s = r\Delta\theta$

Linear vs. Angular Velocity



- Linear speed / velocity given by $\vec{v}$
- Angular speed / velocity given by $\vec{\omega}$
- $\vec{v} = \frac{d\vec{s}}{dt}$
- $\vec{\omega} = \frac{d\vec{\theta}}{dt}$

Direction of Angular Velocity



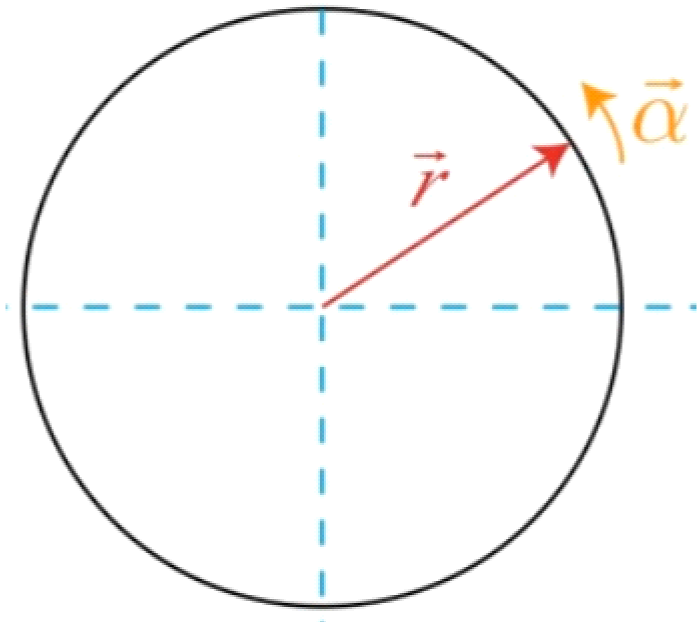
Converting Linear to Angular Velocity

$$v = \frac{ds}{dt} \xRightarrow{s = r\theta} \frac{d r\theta}{dt} \xRightarrow{r \text{ constant}}$$

$$v = r\frac{d\theta}{dt} \xRightarrow{\omega = \frac{d\theta}{dt}}$$

$$\vec{v} = r\vec{\omega}$$

Linear vs. Angular Acceleration



- Linear acceleration is given by $\vec{a}$
- Angular acceleration is given by $\vec{\alpha}$
- $\vec{a} = \frac{d\vec{v}}{dt}$
- $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$

Centripetal Acceleration

- Express position vector in terms of unit vectors.

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} \xrightarrow[y(t)=r\sin\theta]{x(t)=r\cos\theta}$$

$$\vec{r}(t) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} \xrightarrow{\theta=\omega t}$$

$$\vec{r}(t) = r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j}$$

$$v = \frac{dr}{dt} = \frac{d}{dt}\left(r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j}\right) \Rightarrow$$

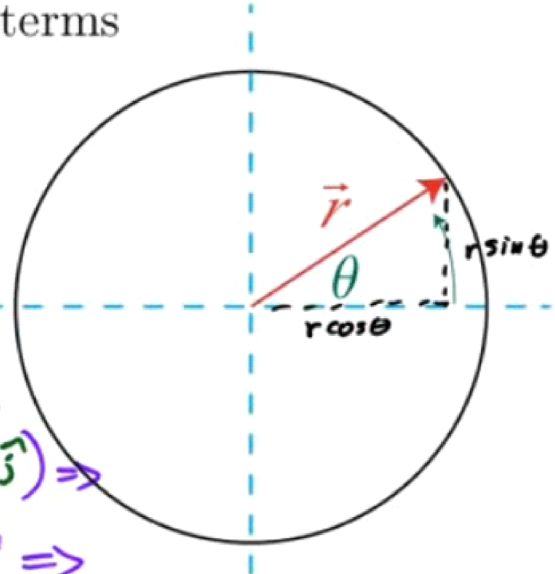
$$v = -\omega r\sin(\omega t)\hat{i} + \omega r\cos(\omega t)\hat{j} \Rightarrow$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(-\omega r\sin(\omega t)\hat{i} + \omega r\cos(\omega t)\hat{j}\right) \Rightarrow$$

$$a = -\omega^2 r\cos(\omega t)\hat{i} - \omega^2 r\sin(\omega t)\hat{j} \Rightarrow$$

$$a = -\omega^2\left(r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j}\right) \xrightarrow{r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j} = \vec{r}}$$

$$a = -\omega^2 r \qquad |a| = \omega^2 r \xrightarrow[\omega=\frac{v}{r}]{v=\omega r} |a| = \frac{v^2}{r^2}r = \frac{v^2}{r}$$



Reference Frames

- A reference frame describes the motion of an observer
  - Most common reference frame is Earth
- Laws of physics we study in this course assume we're in an inertial, non-accelerating reference frame
- There is no way to distinguish between motion at rest and motion at a constant velocity in an inertial reference frame

Calculating Relative Velocities

- Consider two objects, A and B.
- Calculating the velocity of A with respect of reference frame B (and vice versa) is straightforward
- Example:
  - Speed of car with respect to the ground
  - Walking on a train, speed of a person with respect to the train
- $v_{\text{A with respect to C}} = v_{\text{A with respect to B}} + v_{\text{B with respect to C}}$

Linear vs. Angular

Rotational Motion (α = constant)	Linear Motion (a = constant)
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \frac{1}{2}(\omega_0 + \omega)t$	$x = \frac{1}{2}(v_0 + v)t$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$

3.1 - Newton's First Law & Free Body Diagrams

Monday, March 13, 2017 10:15 AM

Newton's First Law of Motion

- An object at rest will remain at rest, and an object in motion will remain in motion, at constant velocity and in a straight line, un line acted upon by a net force
- An object will continue in its current state of motion unless an unbalanced force acts upon it
- An object at rest will remain at rest unless an unbalanced force acts upon them
- Also known as the law of inertia

Force

- A force is a push or pull on an object
- Unites of force are newtons (N)
- $1\text{N} = 1\frac{\text{km} \times \text{m}}{s^2}$

Contact Force	Field Force
Tension Applied Force Friction	Gravity Electrical Force Magnetic Force

Net Force

- A net force is the vector sum of all the forces acting on an object
- If all forces are balanced, there is no net force. This situation is known as translational equilibrium
- An unbalanced force is a net force

Equilibrium

- Static Equilibrium
  - Net force on an object is 0
  - Net torque on an object is 0

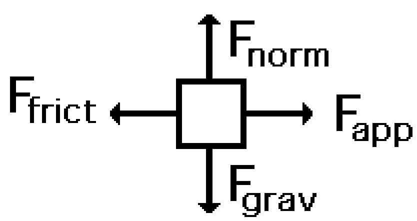
    - Object is at rest
- Mechanical Equilibrium
    - Net force on an object is 0
    - Net torque on an object is 0
- Translational Equilibrium
    - Net force on an object is 0

Inertia

- Inertia is the tendency of an object to resist a change in velocity
- Mass actually has two aspects
    - Inertial mass is how hard it is to change an object's velocity
    - Gravitational mass is how strongly a gravitational field affects a mass
- For the purposes of basic introductory physics, mass and inertia are synonymous

Free Body Diagrams

- Tools used to analyze physical situations
- Show all the force acting on a single object
- Object itself drawn as a dot or rectangle



3.2 - Newton's Second & Third Laws of Motion

Monday, March 13, 2017 10:53 AM

Newton's Second Law of Motion

- The acceleration of an object is in the direction of and directly proportional to the net force applied, and inversely proportional to the object's mass
- Valid only in inertial reference frames.
- $\vec{F}_{net} = \sum \vec{F} = m\vec{a}$

Mass vs. Weight

- Mass is the amount of "stuff" something is made up of.
- It remains constant
- Weight (mg) is the force of gravity on an object
    - Weight varies with gravitational field strength (g)

Newton's Third Law of Motion

- All forces come in pairs. If Object 1 exerts a force on Object 2, then the Object 2 must exert a force back on Object 1, which is equal in magnitude and opposite in direction
- $\vec{F}_{1\ on\ 2} = -\vec{F}_{2\ on\ 1}$

3.3 - Friction

Monday, March 13, 2017 11:02 AM

Coefficient of Friction

Approximate Coefficients of Friction

	Kinetic	Static
Rubber on concrete (dry)	0.68	0.90
Rubber on concrete (wet)	0.58	
Rubber on asphalt (dry)	0.67	0.85
Rubber on asphalt (wet)	0.53	
Rubber on ice	0.15	
Waxed ski on snow	0.05	0.14
Wood on wood	0.30	0.42
Steel on steel	0.57	0.74
Copper on steel	0.36	0.53
Teflon on Teflon	0.04	

- Ratio of the frictional force and the normal force provides the coefficient of friction
- $\mu = \frac{F_f}{F_N}$

Kinetic or Static

Sled sliding down a snowy hill Kinetic

Refrigerator at rest that you want to move static

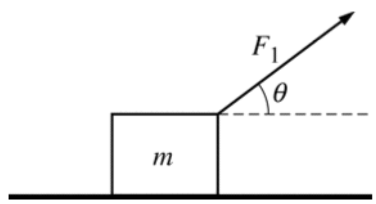
Car with tires rolling freely? static

Car skidding across pavement? Kinetic

Calculating the Force of Friction

- The force of friction depends only upon the nature of the surface in contact (μ) and magnitude of the normal force F_N
- Combine with Newton's Second Las and FBDs to solve more involved problems

2007 Free Response Question 1



Mech. 1.

A block of mass m is pulled along a rough horizontal surface by a constant applied force of magnitude F_1 that acts at an angle θ to the horizontal, as indicated above. The acceleration of the block is a_1. Express all algebraic answers in terms of m, F_1, θ, a_1, and fundamental constants.

(a) On the figure below, draw and label a free-body diagram showing all the forces on the block.

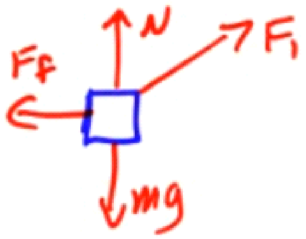
(b) Derive an expression for the normal force exerted by the surface on the block.

(c) Derive an expression for the coefficient of kinetic friction μ between the block and the surface.

(d) On the axes below, sketch graphs of the speed v and displacement x of the block as functions of time t if the block started from rest at $x = 0$ and $t = 0$.

(e) If the applied force is large enough, the block will lose contact with the surface. Derive an expression for the magnitude of the greatest acceleration a_{max} that the block can have and still maintain contact with the ground.

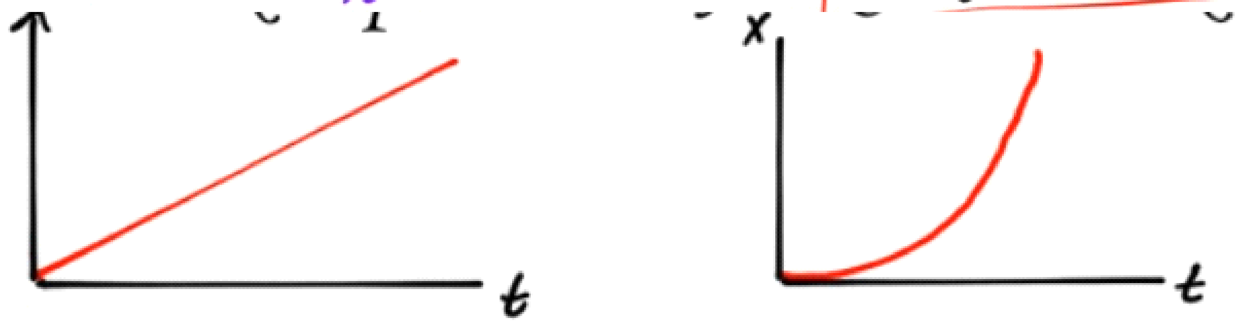
a)



b) $F_{NET_y} = N + F_1 \sin\theta - mg = 0 \Rightarrow \boxed{N = mg - F_1 \sin\theta}$

c) $F_{NET_x} = F_1\cos\theta - F_f \xrightarrow{F_f = \mu F_N} F_1\cos\theta - \mu N = ma_1 \Rightarrow \mu N = F_1\cos\theta - ma$

$$\Rightarrow \mu = \frac{F_1\cos\theta - ma_1}{N} \xrightarrow{N = mg - F_1\sin\theta} \boxed{\mu = \frac{F_1\cos\theta - ma_1}{mg - F_1\sin\theta}}$$



e) $F_N = 0 \Rightarrow F_1 \sin\theta = mg$

$F_1 = \dfrac{mg}{\sin\theta}$ ①

$F_{NET_x} = F_1\cos\theta = ma$

$a = \dfrac{F_1\cos\theta}{m}$ ②

①+② $$a = \frac{F_1\cos\theta}{m} = \frac{\cancel{m}g\cos\theta}{\sin\theta\ \cancel{m}} = \boxed{\frac{g}{\tan\theta}}$$

3.4 - Retarding & Drag Forces

Monday, March 13, 2017 11:12 AM

Retarding Forces

- Sometimes the frictional force is a function of an object's velocity (such as air resistance)
- These forces are called drag, or retarding, forces.

The Skydiver

- Assume we drop Alex from an airplane
- Typically the drag forces on a free-falling object take the form $F_{drag} = bv$ or $F_{drag} = cv^2$, where b and c are constants
- For this problem, let's assume $F_{drag} = bv$

$$F_{NET_y} = mg - F_{drag} = ma_y \xrightarrow{F_{drag}=bv} mg - bv = ma$$

- Initially @ $t=0$, $v=0$, therefore $a=g$

as $t\uparrow$, eventually Alex reaches a max, or terminal velocity

v_t and $a=0$, $F_{drag} = mg$

$$mg - bv = ma \xrightarrow[v>v_t]{a=0} mg - bv_t = 0 \Rightarrow \boxed{v_t = \frac{mg}{b}}$$

Velocity as a Function of Time

$$mg - bv = ma \xrightarrow{a = dv/dt} mg - bv = m\frac{dv}{dt} \Rightarrow \frac{mg}{b} - v = \frac{m}{b}\frac{dv}{dt} \xrightarrow{v_t = \frac{mg}{b}}$$

$$v_t - v = \frac{m}{b}\frac{dv}{dt} \Longrightarrow \frac{dv}{v_t - v} = \frac{b}{m}dt \Rightarrow \frac{dv}{v - v_t} = \frac{-b}{m}dt \Longrightarrow$$

$$\int_{v=0}^{v}\frac{dv}{v - v_t} = \int_{t=0}^{t} -\frac{b}{m}dt \xrightarrow[\int \frac{du}{u} = \ln u + C]{u = v - v_t,\ du = dv} \ln(v - v_t)\Big|_0^v = -\frac{b}{m}t \Longrightarrow$$

$$\ln(v - v_t) - \ln(-v_t) = -\frac{b}{m}t \xrightarrow{\ln a - \ln b = \ln\frac{a}{b}} \ln\left(\frac{v - v_t}{-v_t}\right) = \frac{-b}{m}t \Rightarrow$$

$$\ln\left(\frac{v_t - v}{v_t}\right) = \frac{-b}{m}t \Rightarrow e^{\ln\left(1 - \frac{v}{v_t}\right)} = e^{-\frac{b}{m}t} \Rightarrow 1 - \frac{v}{v_t} = e^{-\frac{b}{m}t} \Longrightarrow$$

$$\frac{v}{v_t} = 1 - e^{-\frac{b}{m}t} \Rightarrow \boxed{v = v_t\left(1 - e^{-\frac{b}{m}t}\right)} \xrightarrow{v_t = \frac{mg}{b}} \boxed{v = \frac{mg}{b}\left(1 - e^{-\frac{b}{m}t}\right)}$$

Acceleration as a Function of Time

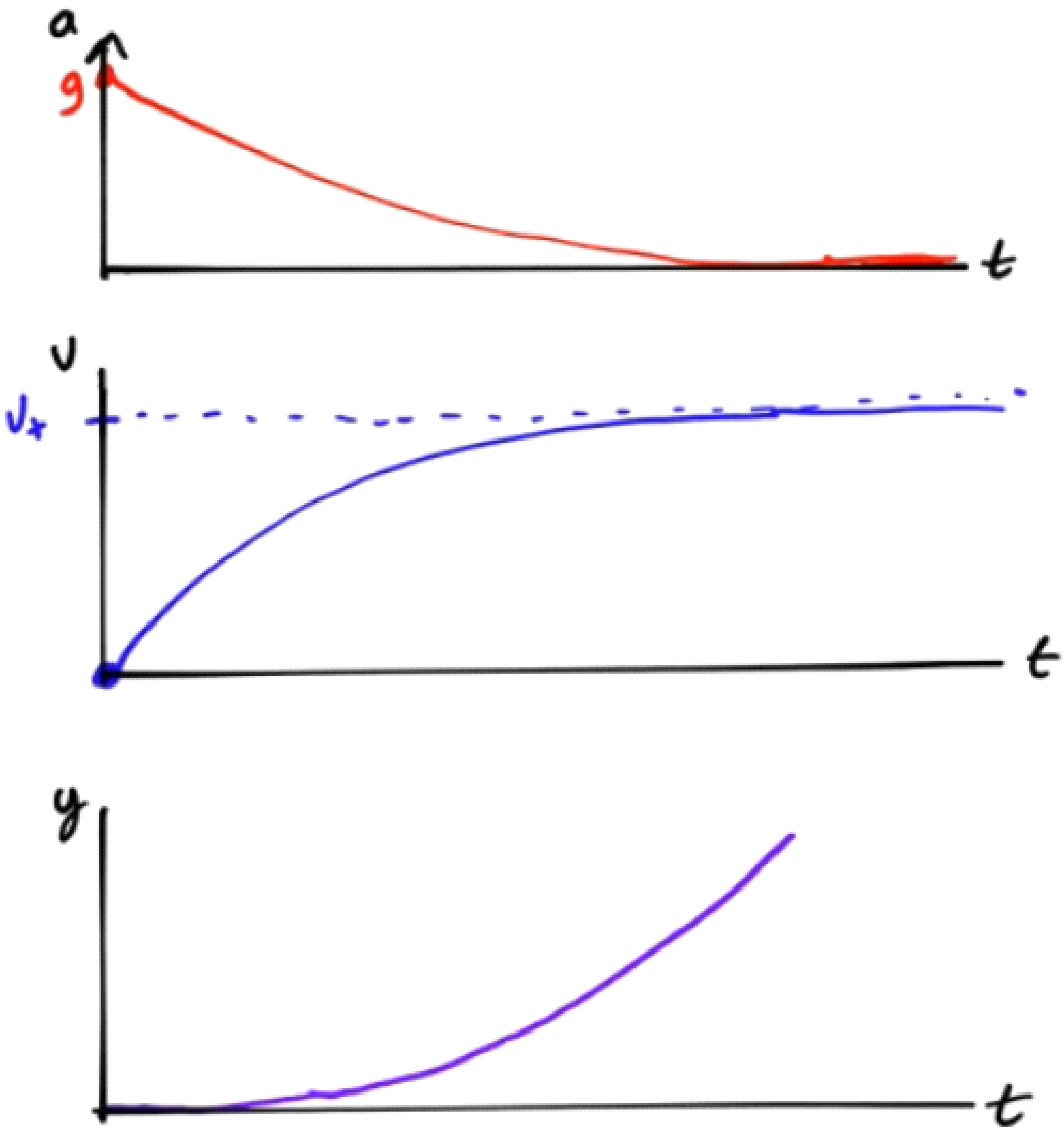
$$a = \frac{dv}{dt} = \frac{d}{dt}\left(v_t - v_t e^{-\frac{b}{m}t}\right) = \frac{d}{dt}\left(-v_t e^{-\frac{b}{m}t}\right) = -v_t \frac{d}{dt}\left(e^{-\frac{b}{m}t}\right) \Rightarrow$$

$$a = -v_t e^{-\frac{b}{m}t}\left(-\frac{b}{m}\right) \Rightarrow a = \cancel{-}\left(\frac{\cancel{m}g}{\cancel{b}}\right)\left(\cancel{+}\frac{\cancel{b}}{\cancel{m}}\right)e^{-\frac{b}{m}t} \Rightarrow$$

$$\boxed{a = g e^{-\frac{b}{m}t}}$$

$$1 - e^{-\frac{t}{\tau}} \qquad e^{-\frac{t}{\tau}}$$

Graph of Acceleration, Velocity, and Displacement



2005 Free Response Question 1

A ball of mass M is thrown vertically upward with an initial speed of v_0. It experiences a force of air resistance given by $\mathbf{F} = -k\mathbf{v}$, where k is a positive constant. The positive direction for all vector quantities is upward. Express all algebraic answers in terms of M, k, v_0, and fundamental constants.

(a) Does the magnitude of the acceleration of the ball increase, decrease, or remain the same as the ball moves upward?

____ increases ____ decreases ____ remains the same

Justify your answer.

(b) Write, but do NOT solve, a differential equation for the instantaneous speed v of the ball in terms of time t as the ball moves upward.

(c) Determine the terminal speed of the ball as it moves downward.

(d) Does it take longer for the ball to rise to its maximum height or to fall from its maximum height back to the height from which it was thrown?

____longer to rise ____longer to fall

Justify your answer.

(e) On the axes below, sketch a graph of velocity versus time for the upward and downward parts of the ball's flight, where t_f is the time at which the ball returns to the height from which it was thrown.

a) $\uparrow +y$ $\quad \downarrow mg \quad \downarrow kv$ $\quad F_{NET,y} = -mg - kv = ma_y \Rightarrow a_y = -g - \frac{kv}{m}$

as $v\downarrow$ $\quad |a_y|$ $\boxed{\text{decrease}}$

b) $a_y = -g - \frac{kv}{m} \xRightarrow{a=\frac{dv}{dt}} \frac{dv}{dt} = -g - \frac{kv}{m} \Rightarrow M\frac{dv}{dt} = -Mg - kv$

c) @ $V_{term} \Rightarrow F_{NET} = 0 \Rightarrow$ $\uparrow kv$ $\downarrow Mg$ $\Rightarrow kV_t = Mg \Rightarrow \boxed{V_t = \frac{Mg}{K}}$

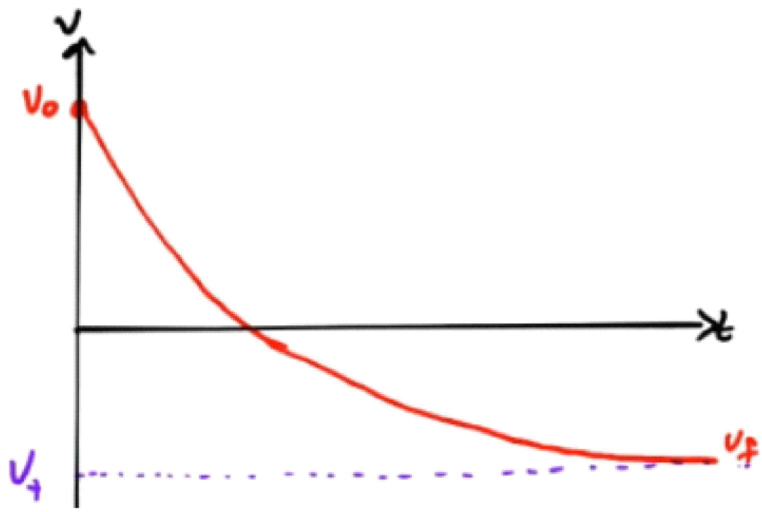
d) On way up, friction brings ball to a stop quickly

On way down, friction slows ball down

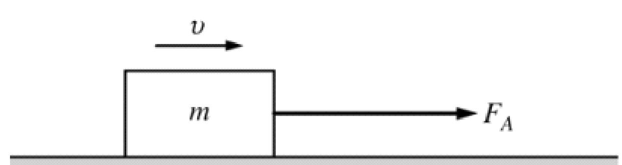
$\bar{v}_{up} > \bar{v}_{down}$, d is constant, so $t_{down} > t_{up}$ $t = \frac{d}{v}$

$\boxed{\text{longer to fall}}$

e)



2013 Free Response Question 2



Mech 2.

A box of mass m initially at rest is acted upon by a constant applied force of magnitude F_A, as shown in the figure above. The friction between the box and the horizontal surface can be assumed to be negligible, but the box is subject to a drag force of magnitude kv where v is the speed of the box and k is a positive constant. Express all your answers in terms of the given quantities and fundamental constants, as appropriate.

(a) The dot below represents the box. Draw and label the forces (not components) that act on the box.

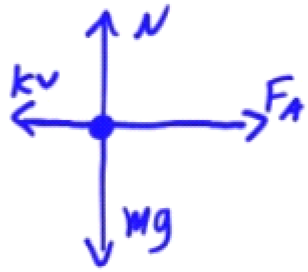
(b) Write, but do not solve, a differential equation that could be used to determine the speed v of the box as a function of time t. If you need to draw anything other than what you have shown in part (a) to assist in your solution, use the space below. Do NOT add anything to the figure in part (a).

(c) Determine the magnitude of the terminal velocity of the box.

(d) Use the differential equation from part (b) to derive the equation for the speed v of the box as a function of time t. Assume that $v = 0$ at time $t = 0$.

(e) On the axes below, sketch a graph of the speed v of the box as a function of time t. Explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expressions, as appropriate.

a)



b) $F_{net_x} = F_A - kv = ma \xrightarrow{a=\frac{dv}{dt}} \boxed{F_A - kv = m\frac{dv}{dt}}$

c) @ v_{term}, $a=0$, $F_{net}=0$, $\Rightarrow F_A = kv \Rightarrow F_A = kv_t \Rightarrow \boxed{v_t = \frac{F_A}{k}}$

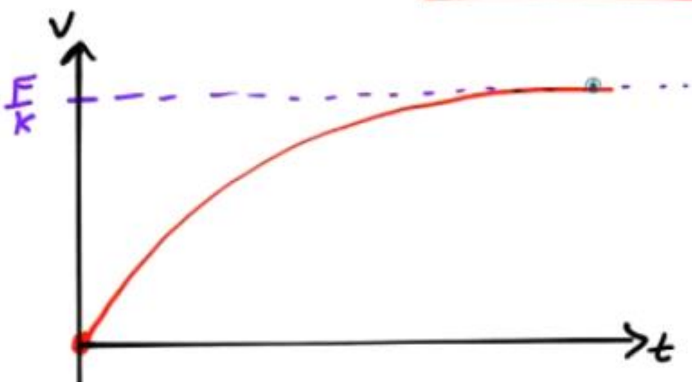
d) $F - kv = m\frac{dv}{dt} \Rightarrow \frac{dv}{F-kv} = \frac{dt}{m} \Rightarrow -\frac{1}{k}\int_{v=0}^{v}\frac{-k\,dv}{F-kv} = \int_{t=0}^{t}\frac{dt}{m} \Rightarrow$

$\ln(F-kv)\Big|_0^v = \frac{-kt}{m} \Rightarrow \ln(F-kv) - \ln(F) = \frac{-kt}{m} \Rightarrow$

$e^{\ln\left(\frac{F-kv}{F}\right)} = e^{\frac{-kt}{m}} \Rightarrow \frac{F-kv}{F} = e^{-\frac{k}{m}t} \Rightarrow F-kv = Fe^{-\frac{k}{m}t} \Rightarrow$

$kv = F - Fe^{-\frac{k}{m}t} \Rightarrow \boxed{v = \frac{F}{k}\left(1-e^{-\frac{k}{m}t}\right)}$

e)

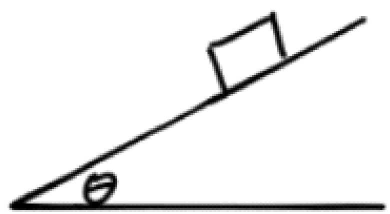


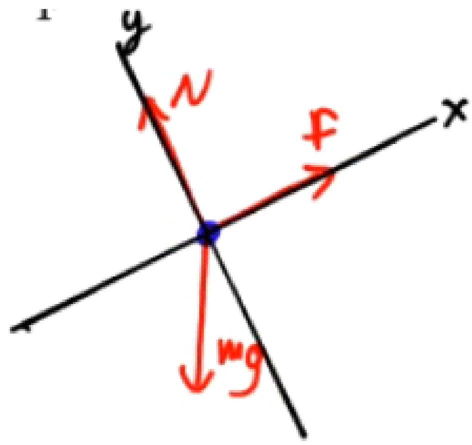
3.5 - Ramps & Inclines

Tuesday, March 14, 2017 9:35 PM

Drawing FBDs for Ramps

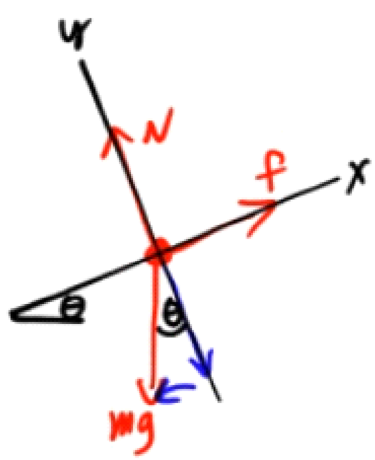
- Choose the object of interest and draw it as a dot or box
- Draw and label all the external forces acting on the object
- Sketch a coordinate system, choosing the direction of the object's motion as one of the positive axes

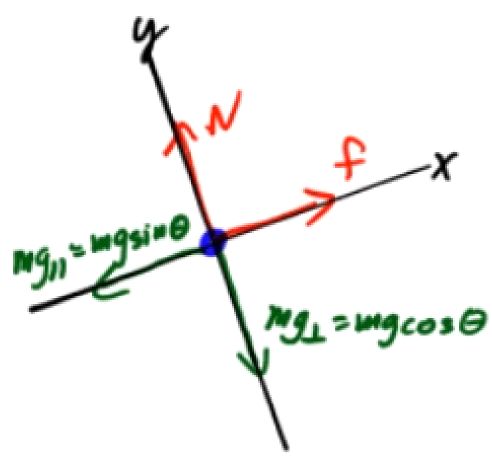




Pseudo-FBDs

- When forces don't line up with axes, you can draw a pseudo-FBD
- Break forces that don't line up with axes into components that do
- Redraw your diagram with all forces parallel to axes





3.6 - Atwood Machines

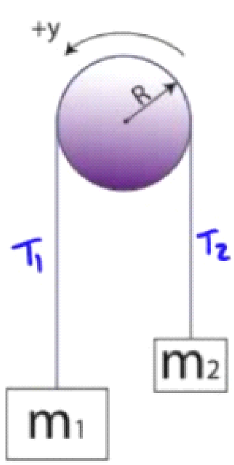
Tuesday, March 14, 2017 9:44 PM

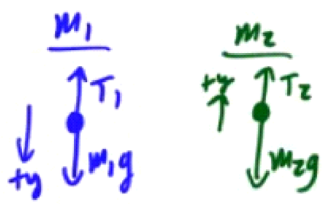
Atwood Machine

- Atwood Machine - two objects are connected by a light string over a massless pulley
- Ideal pulleys are frictionless and massless (they add no inertia to the system)
- Tension is constant in a light string passing over an ideal pulley

Solving Atwood Machine Problems

1. Adopt a sign convention for positive and negative motion
2. Draw a free body diagram for each mass
3. Write Newton's Second Law equations for each mass
4. Solve for unknowns





$$m_1 g - T_1 = m_1 a$$

$$T_2 - m_2 g = m_2 a$$

$$m_1 g \underbrace{- T_1 + T_2}_{0} - m_2 g = a(m_1 + m_2)$$

$$m_1 g - m_2 g = a(m_1 + m_2)$$

$$g(m_1 - m_2) = a(m_1 + m_2)$$

$$\boxed{a = g\left(\frac{m_1 - m_2}{m_1 + m_2}\right)}$$

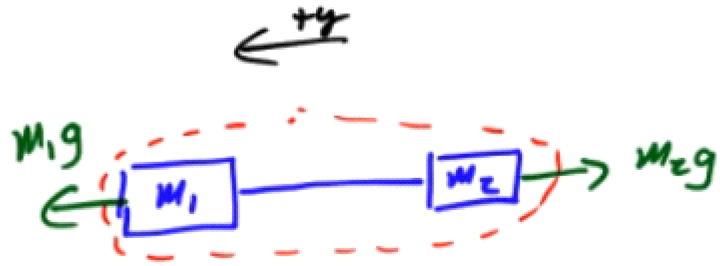
$$m_1 : F_{net_y} = m_1 g - T_1 = m_1 a$$

$$m_2 : F_{net_y} = T_2 - m_2 g = m_2 a$$

$$T_1 = T_2$$

Alternate Solution

- Analyze the system as a whole

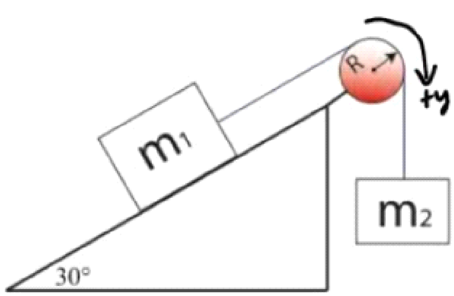


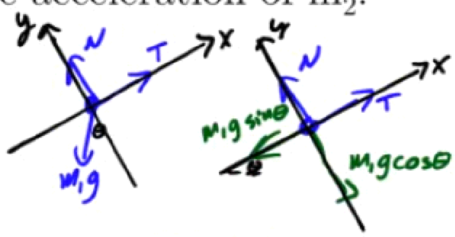
$$F_{NET_y} = m_1 g - m_2 g = (m_1 + m_2) a$$

$$\boxed{a = g\frac{(m_1 - m_2)}{(m_1 + m_2)}}$$

Mass and Pulley on a Ramp

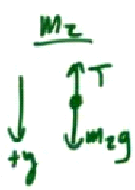
- Two masses, m_1 and m_2, are connected by a light string over a massless pulley as shown. Assuming a frictionless surface, find the acceleration of m_2.





$$T - m_1 g \sin 30^\circ = m_1 a$$

$$T = m_1 g \sin 30^\circ + m_1 a$$



$$m_2 g - T = m_2 a$$

$$m_2 g - (m_1 g \sin 30^\circ + m_1 a) = m_2 a$$

$$m_2 g - m_1 g \sin 30^\circ - m_1 a = m_2 a$$

$$g(m_2 - m_1 \sin 30^\circ) = a(m_1 + m_2) \Rightarrow a = g\frac{(m_2 - m_1 \sin 30^\circ)}{(m_1 + m_2)}$$

4.1 - Work

Tuesday, March 14, 2017 9:55 PM

What is Work?

- Work is the process of moving an object by applying a force
- The object must move for work to be done
- The force must cause the movement
- Work is a scalar quantity
- Units are Joules

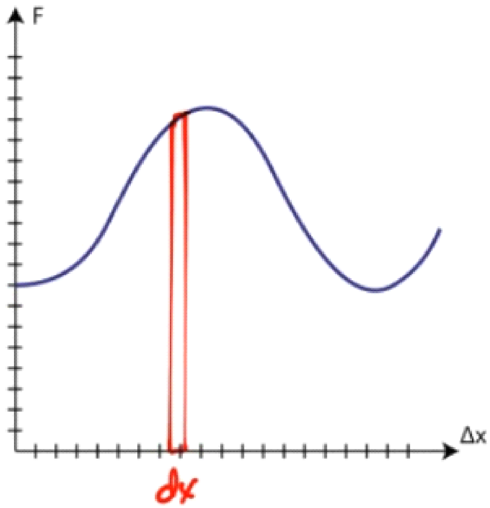
Work in One Dimension

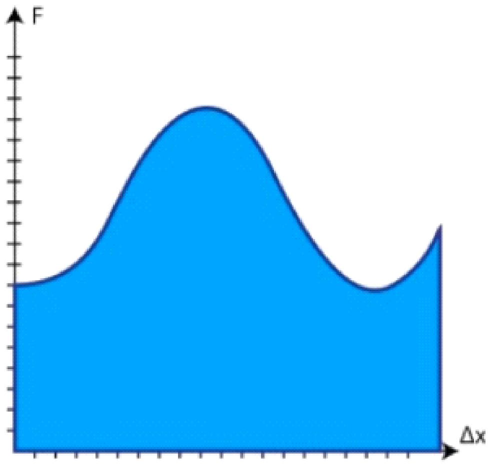
- Only the force in the direction of the displacement contributes to the work done

$$W = F\cos\theta\,\Delta x = \vec{F}\cdot\Delta\vec{x}$$

Non-Constant Forces

- Work done is the area under the force vs. displacement graph

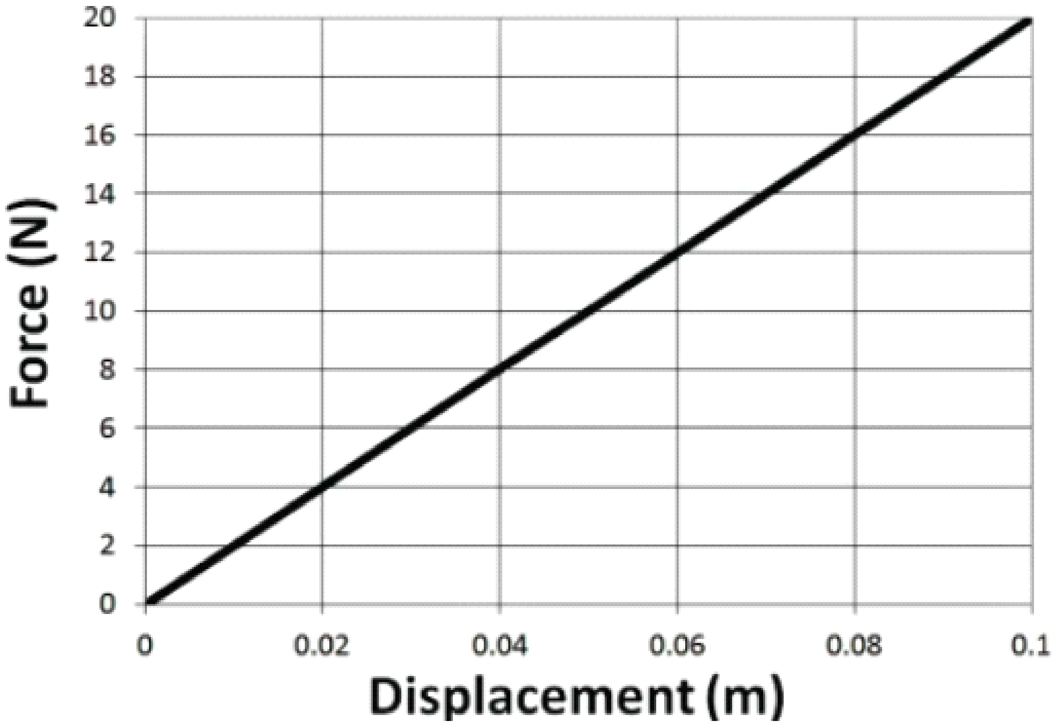




$$W = \int_{x_i}^{x_f} F(x)\,dx$$

Hooke's Law

- The more you stretch or compress a string, the greater the force of the spring
- The spring's force is opposite the direction of its displacement from equilibrium
- Model this as a linear relationship, where the force applied by the spring is equal to a constant (the spring constant) multiplied by the spring's displacement from its equilibrium (rest) position
- $F_s = -kx$
- Slope of the graph gives you the spring constant, k (in N/m)



Work Done in Compressing a Spring

- A spring obeys Hooke's Law. How much work is done in compressing the spring from equilibrium to some point x?

$$W = \int \vec{F} \cdot d\vec{x} = \int_{x=0}^{x} kx\,dx \Rightarrow W = k\int_0^x x\,dx \Rightarrow$$

$$W = k\,\frac{x^2}{2}\Big|_0^x = k\left(\frac{x^2}{2} - 0\right) = \frac{1}{2}kx^2 \Rightarrow$$

$$U_s = \frac{1}{2}kx^2$$

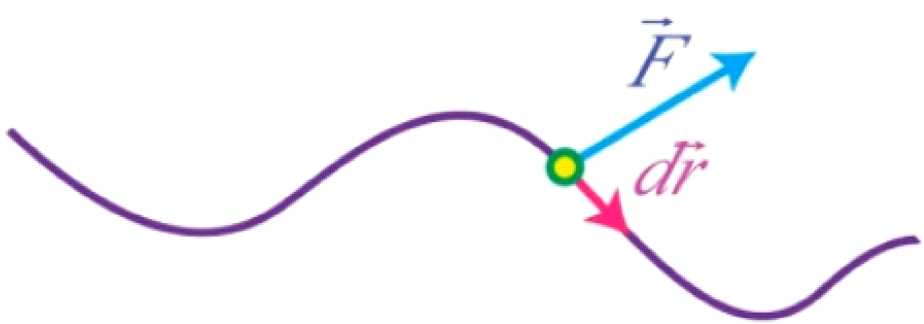
Work Done in Compressing a Non-Linear Spring

The force required to extend a non-linear spring is described by $F(x)=0.5kx^2$. How much work is done in compressing the spring from equilibrium to some point x?

$$W = \int_0^x \vec{F} \cdot d\vec{x} = \int_0^x \frac{1}{2}kx^2\,dx = \frac{k}{2}\int_0^x x^2\,dx = \frac{k}{2}\,\frac{x^3}{3}\Big|_0^x \Rightarrow$$

$$W = \frac{k}{2}\left(\frac{x^3}{3} - \frac{0^3}{3}\right) = \frac{kx^3}{6}$$

Work in Multiple Dimensions



$$W = \int dW = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

Work-Energy Theorem

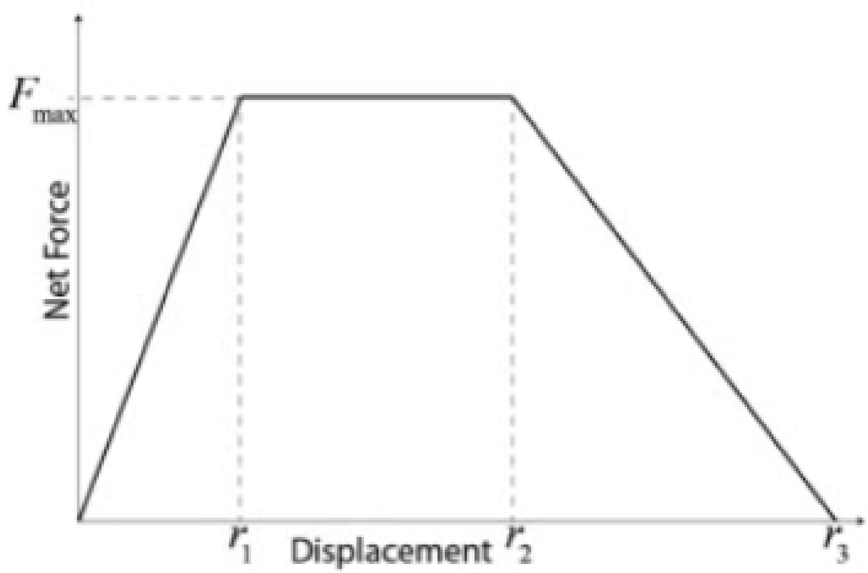
$$W \int_{x_i}^{x_f} F(x)\,dx \xrightarrow[v=\frac{dx}{dt} \Rightarrow dx = v\,dt]{F=ma=m\frac{dv}{dt}} W = \int_{v_i}^{v_f} m\frac{dv}{\cancel{dt}}\, v\,\cancel{dt} \Longrightarrow$$

$$W = \int_{v_i}^{v_f} mv\,dv = m\int_{v_i}^{v_f} v\,dv = m\frac{v^2}{2}\Big|_{v_i}^{v_f} = m\left(\frac{v_f^2}{2} - \frac{v_i^2}{2}\right)$$

$$\xrightarrow{K=\frac{1}{2}mv^2} W = K_f - K_i = \Delta K$$

$$\boxed{W = \Delta K}$$

Example: Velocity from an F-d Graph



- Determine the object's final speed.

$$W = \text{Area} = \frac{1}{2}bh + lw + \frac{1}{2}bh \Rightarrow$$

$$\frac{1}{2}r_1 F_{max} + (r_2 - r_1)F_{max} + \frac{1}{2}(r_3 - r_2)F_{max} = \frac{1}{2}mv^2 \Rightarrow$$

$$v^2 = \frac{F_{max}}{m}(r_3 + r_2 - r_1) \Rightarrow$$

$$v = \sqrt{\frac{F_{max}}{m}(r_3 + r_2 - r_1)}$$

4.2 - Energy & Conservative Forces

Tuesday, March 14, 2017 10:10 PM

What is Energy?

- Energy is the ability or capacity to do work
  - Work is the process of moving an object
- Energy is the ability or capacity to move an object

Energy Transformations

- Energy can be transformed from one type to another
- You can transfer energy from one object to another by doing work
- Work-Energy Theorem
  - Work done on a system by an external force changes the energy of the system

Kinetic Energy

- Kinetic Energy is energy of motion
  - The ability or capacity of a moving object to move another object
- $K = \frac{1}{2}mv^2$

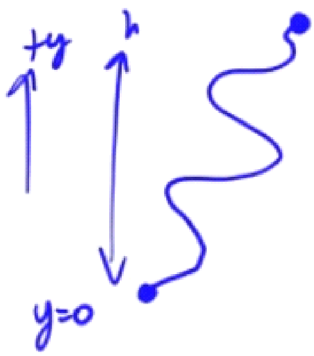
Potential Energy

- Potential Energy (U) is energy an object possesses due to its position or state of being
  - Gravitational Potential Energy (U_g) is the energy an object possesses because of its position in a gravitational field
  - Elastic Potential Energy (U_s)
  - Chemical Potential Energy
  - Electric Potential Energy
  - Nuclear Potential Energy
- A single object can have only kinetic energy, as potential energy requires an interaction between objects

Internal Energy

- The internal energy of a system include the kinetic energy of the objects that make up the system and the potential energy of the configuration of the objects that make up the system
- Changes in a system's structure can result in changes in internal energy

Gravitational Potential Energy (U_g)



$$W = \int_{y=0}^{y=h} \vec{F}\cdot d\vec{r} = \int_0^h mg\,dy \Rightarrow$$

$$W = mg\int_0^h dy = mgh$$

$$U_g = mgh$$

Conservative Forces

- A force in which the work done on an object is independent of the path taken is known as conservative force
- A force in which the work done moving along a closed path is zero
- A force in which the work done is directly related to a negative change in potential energy (W=-ΔU)

Conservative Forces	Non-Conservative Forces
Gravity Elastic Forces Coulombic Forces	Friction Drag Air Resistance

Newton's Law of Universal Gravitation

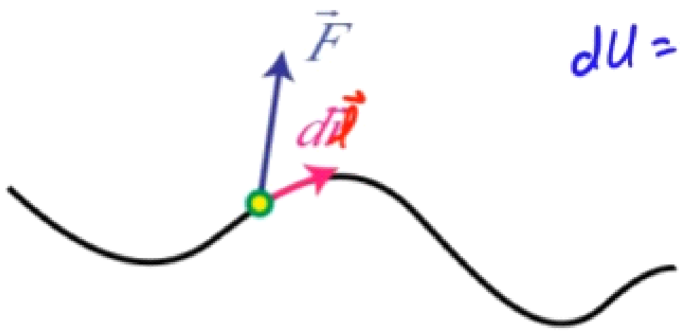
- The gravitational force of attraction between any two objects with mass
- $F_g = -\frac{Gm_1m_2}{r^2}\hat{r}$

$$\Delta U = -W_{cF} = -W_{grav} = -\int_{\infty}^{r} -\frac{Gm_1m_2}{r^2}\,dr \Rightarrow$$

$$U_g = Gm_1m_2 \int_{\infty}^{r} \frac{dr}{r^2} = Gm_1m_2 \int_{\infty}^{r} r^{-2}\,dr = Gm_1m_2\left(\frac{-1}{r}\right)\Big|_{\infty}^{r}$$

$$\boxed{U_g = -\frac{Gm_1m_2}{r}}$$

Force from Potential Energy



$$dU = -dW_F = -\vec{F}\cdot d\vec{\ell} = -\underbrace{F\cos\theta}_{F_\ell}\,d\ell$$

$$dU = -F\cos\theta\,d\ell \Rightarrow dU = -F_\ell\,d\ell$$

$$F_\ell = \frac{-dU}{d\ell}$$

Summary

PE		$F = -dU/dl$
$U_g = mgh$	$\longleftrightarrow$	$F = -mg$
$U_g = \frac{-Gm_1m_2}{r}$	$\longleftrightarrow$	$F = \frac{-Gm_1m_2}{r^2}$
$U_s = \frac{1}{2}kx^2$	$\longleftrightarrow$	$F = -kx$

4.3 - Conservation of Energy

Wednesday, March 15, 2017 11:05 AM

Conservation of Mechanical Energy

- Consider a single conservative force doing work on a closed system
- $\because W_F = \Delta K, \quad W_F = -\Delta U$
- $\therefore \Delta K + \Delta U = 0$
- $\therefore K_i + U_i = K_f + U_f$

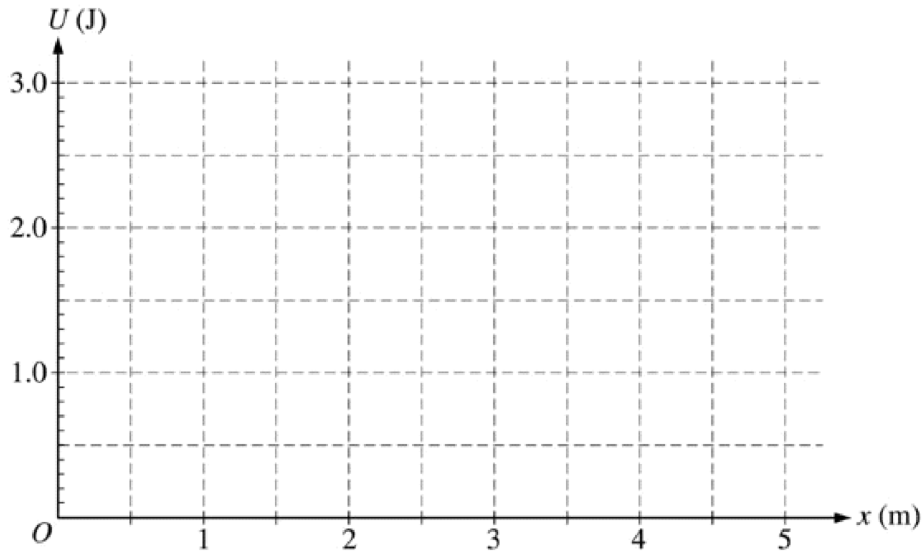
Non-Conservative Forces

- Non-conservative forces change the total mechanical energy of a system, but not the total energy of a system
- Work done by a non-conservative force is typically converted to internal (thermal) energy
- $E_{total} = K + U + W_{NC}$
- $E_{mech} = K + U$

2002 Free Response Question 3

An object of mass 0.5 kg experiences a force that is associated with the potential energy function $U(x) = \frac{4.0}{2.0 + x}$, where U is in joules and x is in meters.

(a) On the axes below, sketch the graph of $U(x)$ versus x.



(b) Determine the force associated with the potential energy function given above.

(c) Suppose that the object is released from rest at the origin. Determine the speed of the particle at $x = 2$ m.

In the laboratory, you are given a glider of mass 0.5 kg on an air track. The glider is acted on by the force determined in part (b). Your goal is to determine experimentally the validity of your theoretical calculation in part (c).

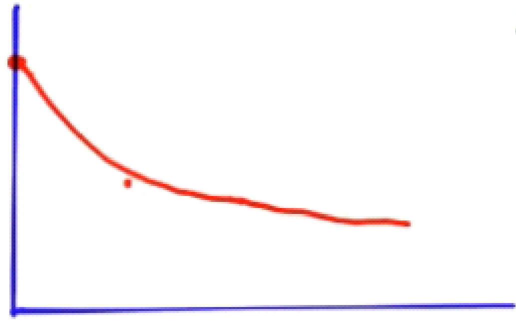
(d) From the list below, select the additional equipment you will need from the laboratory to do your experiment by checking the line next to each item. If you need more than one of an item, place the number you need on the line.

___ Meterstick   ___ Stopwatch   ___ Photogate timer   ___ String   ___ Spring

___ Balance   ___ Wood block   ___ Set of objects of different masses

(e) Briefly outline the procedure you will use, being explicit about what measurements you need to make in order to determine the speed. You may include a labeled diagram of your setup if it will clarify your procedure.

a)



b) $F = -\frac{dU}{d\ell} = -\frac{d}{dx}\left(\frac{4}{2+x}\right) \Rightarrow$

$F = -4\frac{d}{dx}\left(\frac{1}{2+x}\right) = -4\frac{d}{dx}\left((x+2)^{-1}\right)$

$F = 4(x+2)^{-2} = \boxed{\frac{4}{(x+2)^2}}$

c) $\Delta U = U(0) - U(2) = \frac{1}{2}mv^2 \Rightarrow 2-1 = 1J = \frac{1}{2}mv^2 \Rightarrow$

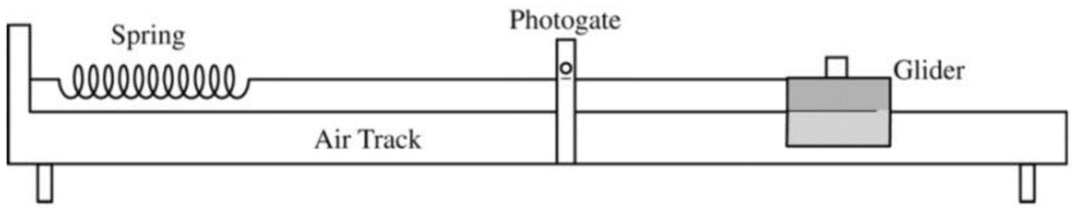
$\frac{2}{m} = v^2 \Rightarrow v^2 = \frac{2}{m} = \frac{2}{.5} = 4 \Rightarrow \boxed{v = 2\ m/s}$

d) Photogate Timer   Meterstick

e)

$\boxed{v = \frac{d}{t}}$

2007 Free Response Question 3



Mech. 3.

The apparatus above is used to study conservation of mechanical energy. A spring of force constant 40 N/m is held horizontal over a horizontal air track, with one end attached to the air track. A light string is attached to the other end of the spring and connects it to a glider of mass m. The glider is pulled to stretch the spring an amount x from equilibrium and then released. Before reaching the photogate, the glider attains its maximum speed and the string becomes slack. The photogate measures the time t that it takes the small block on top of the glider to pass through. Information about the distance x and the speed v of the glider as it passes through the photogate are given below.

Trial #	Extension of the Spring x (m)	Speed of Glider v (m/s)	Extension Squared x^2 (m^2)	Speed Squared v^2 (m^2/s^2)
1	0.30×10^{-1}	0.47	0.09×10^{-2}	0.22
2	0.60×10^{-1}	0.87	0.36×10^{-2}	0.76
3	0.90×10^{-1}	1.3	0.81×10^{-2}	1.7
4	1.2×10^{-1}	1.6	1.4×10^{-2}	2.6
5	1.5×10^{-1}	2.2	2.3×10^{-2}	4.8

(a) Assuming no energy is lost, write the equation for conservation of mechanical energy that would apply to this situation.

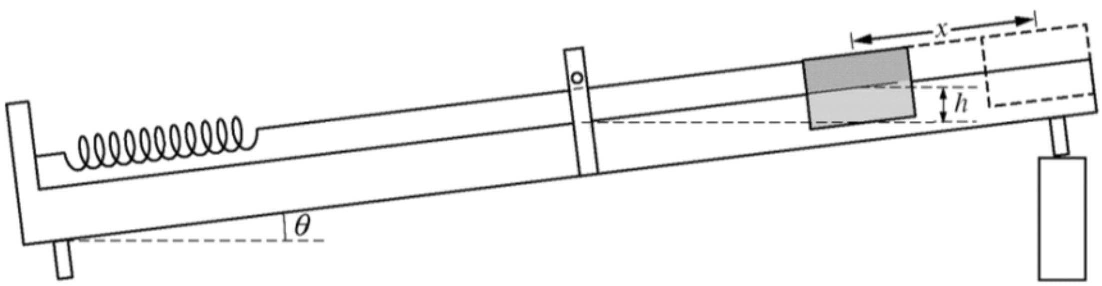
(b) On the grid below, plot v^2 versus x^2. Label the axes, including units and scale.

(c)

i. Draw a best-fit straight line through the data.

ii. Use the best-fit line to obtain the mass m of the glider.

(d) The track is now tilted at an angle θ as shown below. When the spring is unstretched, the center of the glider is a height h above the photogate. The experiment is repeated with a variety of values of x.



i. Assuming no energy is lost, write the new equation for conservation of mechanical energy that would apply to this situation.

ii. Will the graph of v^2 versus x^2 for this new experiment be a straight line?

_____ Yes _____ No

Justify your answer.

a) $K = U_s \Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}kx^2 \xrightarrow{k=40} \frac{1}{2}mv^2 = \frac{1}{2}(40)x^2$

b) v^2 (m^2/s^2) ↑

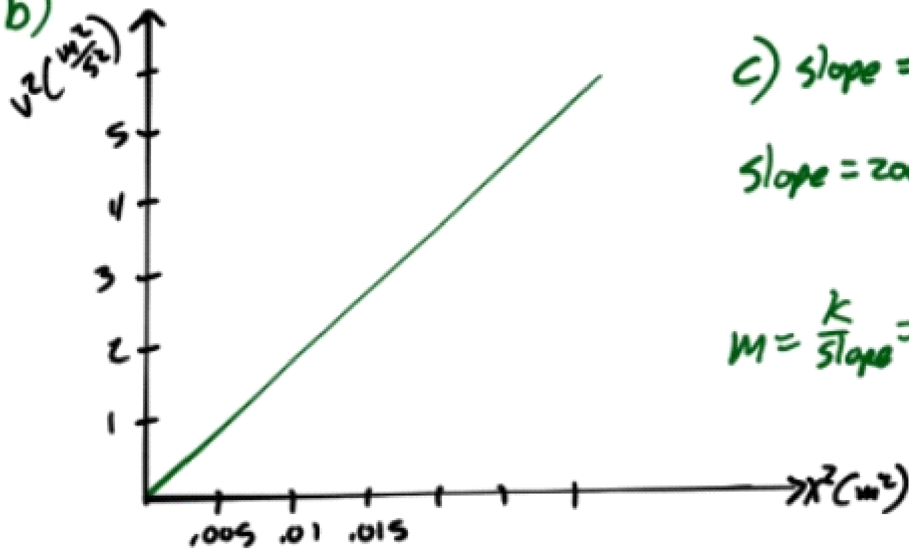
c) slope $= \frac{k}{m}$

$\frac{1}{2}mv^2 = 20x^2$

$mv^2 = 40x^2$

a) $K = U_s \Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}kx^2 \xrightarrow{k=40} \frac{1}{2}mv^2 = \frac{1}{2}(40)x^2$

b)



c) slope $= \frac{k}{m}$

slope $= 200 \frac{1}{s^2}$

$\frac{1}{2}mv^2 = 20x^2$

$mv^2 = 40x^2$

$v^2 = \frac{40}{m}x^2$

$y = m\,x$

$m = \frac{k}{\text{slope}} = \frac{40\,N/m}{200\,\frac{1}{s^2}} = \boxed{0.2\text{ kg}}$

d) $U_s + U_g = K$

$\frac{1}{2}kx^2 + mg(h + x\sin\theta) = \frac{1}{2}mv^2$

NO $\quad v^2$ is a fcn of both x and x^2

2010 Free Response Question 1

Mech. 1.

Students are to conduct an experiment to investigate the relationship between the terminal speed of a stack of falling paper coffee filters and its mass. Their procedure involves stacking a number of coffee filters, like the one shown in the figure above, and dropping the stack from rest. The students change the number of filters in the stack to vary the mass m while keeping the shape of the stack the same. As a stack of coffee filters falls, there is an air resistance (drag) force acting on the filters.

(a) The students suspect that the drag force F_D is proportional to the square of the speed v: $F_D = Cv^2$, where C is a constant. Using this relationship, derive an expression relating the terminal speed v_T to the mass m.

The students conduct the experiment and obtain the following data.

Mass of the stack of filters, m (kg)	1.12×10^{-3}	2.04×10^{-3}	2.96×10^{-3}	4.18×10^{-3}	5.10×10^{-3}
Terminal speed, v_T (m/s)	0.51	0.62	0.82	0.92	1.06

(b)

(i) Assuming the functional relationship for the drag force above, use the grid below to plot a linear graph as a function of m to verify the relationship. Use the empty boxes in the data table, as appropriate, to record any calculated values you are graphing. Label the vertical axis as appropriate, and place numbers on both axes.

(ii) Use your graph to calculate C.

A particular stack of filters with mass m is dropped from rest and reaches a speed very close to terminal speed by the time it has fallen a vertical distance Y.

(c)

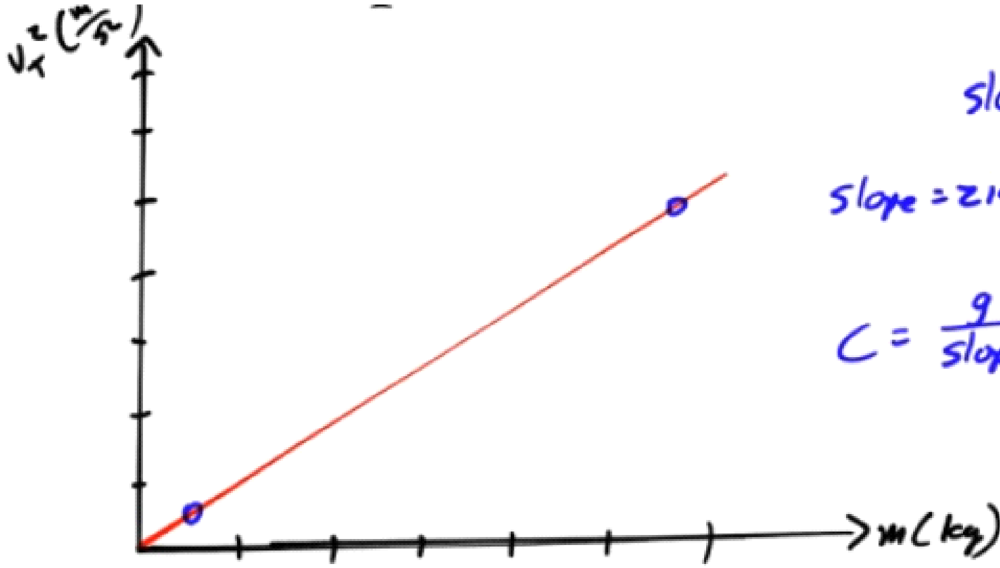
(i) Sketch an approximate graph of speed versus time from the time the filters are released up to the time $t = T$ that the filters have fallen the distance Y. Indicate time $t = T$ and terminal speed $v = v_T$ on the graph.

(ii) Suppose you had a graph like the one sketched in (c)(i) that had a numerical scale on each axis. Describe how you could use the graph to approximate the distance Y.

(d) Determine an expression for the approximate amount of mechanical energy dissipated, ΔE, due to air resistance during the time the stack falls a distance y, where $y > Y$. Express your answer in terms of y, m, v_T, and fundamental constants.

a) 

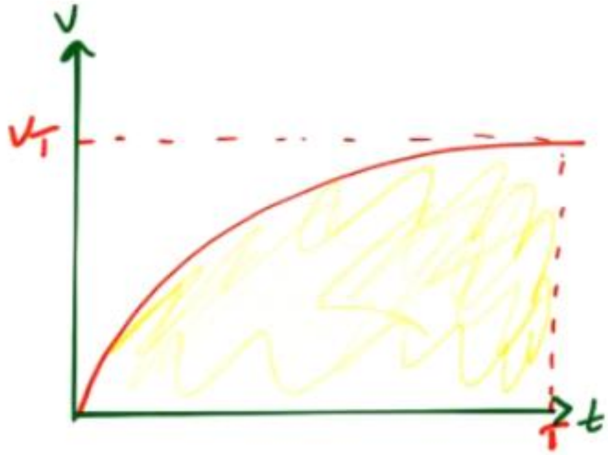
@ $v_{term} \Rightarrow F_{NET_y} = 0 \Rightarrow Cv^2 = mg \Rightarrow v_T = \sqrt{\frac{mg}{c}}$



slope $= \frac{v_T^2}{m} = \frac{g}{c}$

slope $= 215.9 \frac{m^2}{kg\,s^2} = \frac{g}{c} \Rightarrow$

$C = \frac{g}{slope} = .0454 \frac{kg}{m}$

c) 

Cii) Area under graph from $t=0$ to $t=T$ is distance traveled, Y

d) $U_i + K_i = U_f + K_f$

$\Delta E = \mathcal{E}_f - \mathcal{E}_i = \boxed{mgy - \frac{1}{2}mv_T^2}$

2013 Free Response Question 1

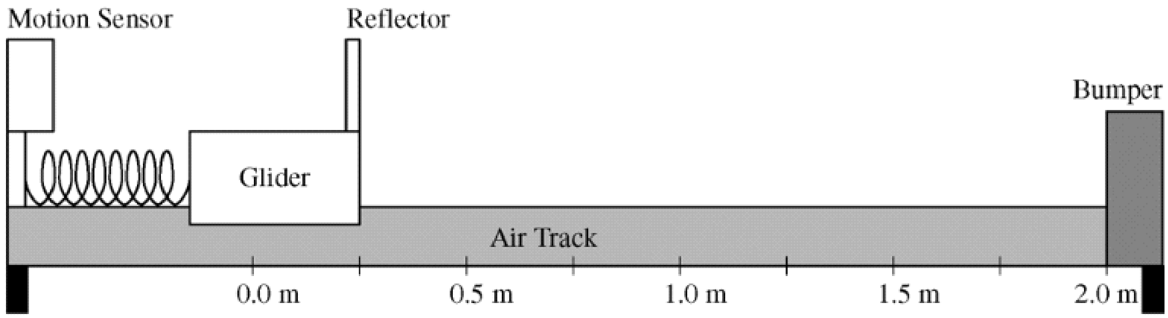


Figure 1

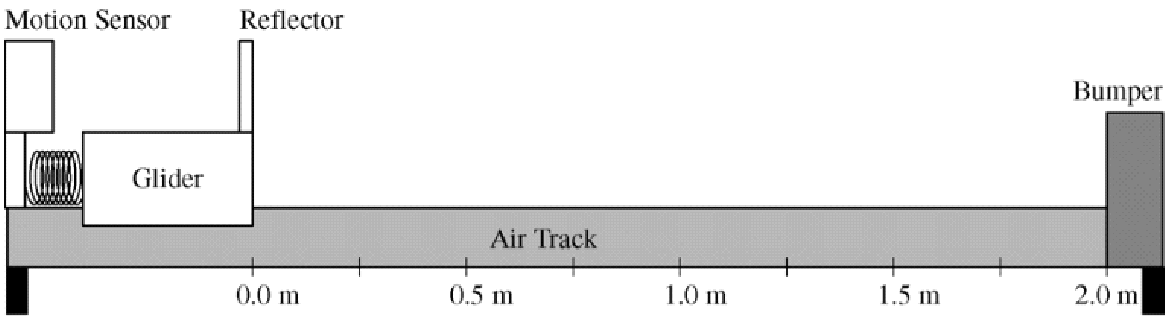


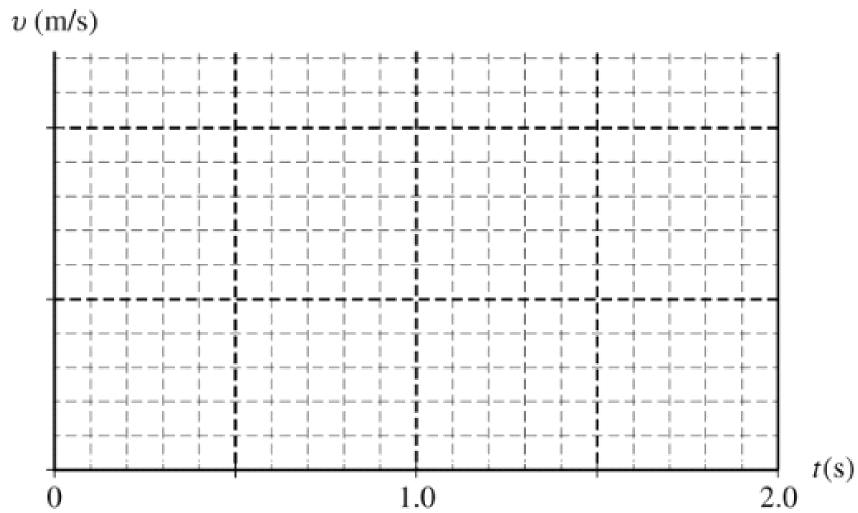
Figure 2

Mech 1.

A student places a 0.40 kg glider on an air track of negligible friction and holds it so that it touches an uncompressed ideal spring, as shown in Figure 1 above. The student then pushes the glider back to compress the spring by 0.25 m, as shown in Figure 2. At time $t = 0$, the student releases the glider, and a motion sensor begins recording the velocity of the reflector at the front of the glider as a function of time. The data points are shown in the table below. At time $t = 0.79$ s, the glider loses contact with the spring.

Time (s)	0	0.25	0.50	0.75	1.00	1.50	2.00
Velocity (m/s)	0	0.25	0.43	0.48	0.50	0.49	0.51

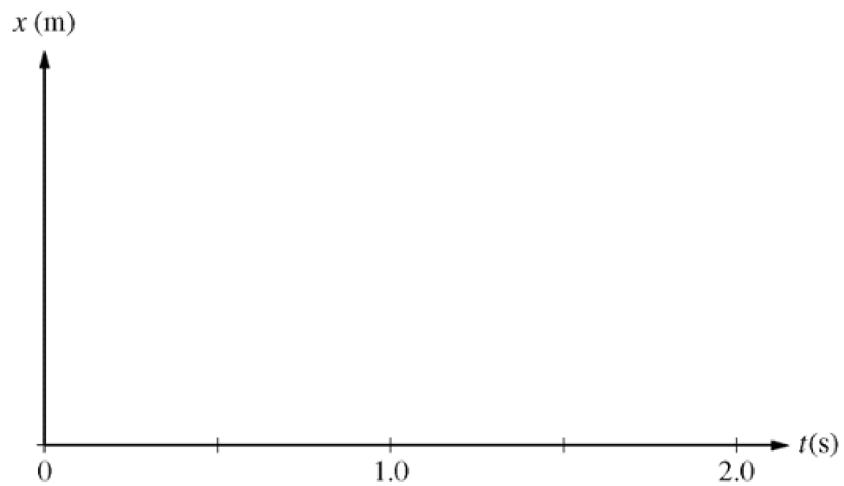
(a) On the axes below, plot the data points for velocity v as a function of time t for the glider, and draw a smooth curve that best fits the data. Be sure to label an appropriate scale on the vertical axis.



(b) The student wishes to use the data to plot position x as a function of time t for the glider.

i. Describe a method the student could use to do this.

ii. On the axes below, sketch the position x as a function of time t for the glider. Explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expressions, as appropriate.



(c) Calculate the time at which the glider makes contact with the bumper at the far right.

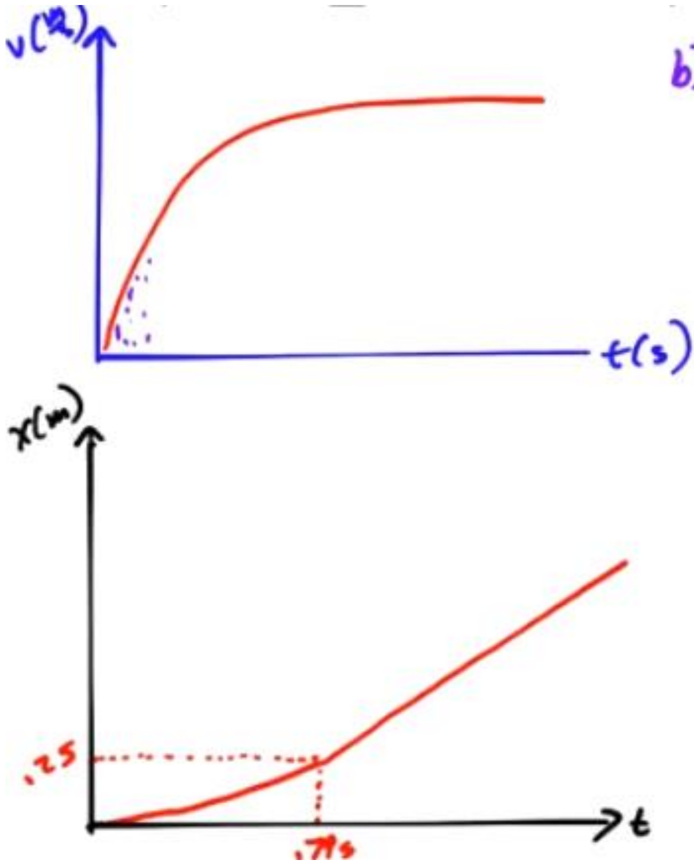
(d) Calculate the force constant of the spring.

(e) The experiment is run again, but this time the glider is attached to the spring rather than simply being pushed against it.

i. Determine the amplitude of the resulting periodic motion.

ii. Calculate the period of oscillation of the resulting periodic motion.

b) i) Plot area under v-t curve as a function of time



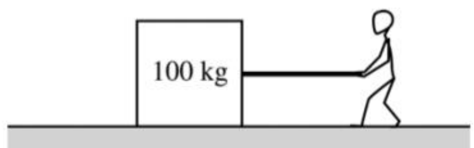
4.4 - Power

Wednesday, March 15, 2017 11:05 AM

Defining Power

- Power is the rate at which work is done
- Power is the rate at which a force does work
- Units of power are joules/second, or watts
- $P_{avg} = \frac{\Delta W}{\Delta t}$
- $P = \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{r}}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$

2003 Free Response Question 1



Mech. 1.

The 100 kg box shown above is being pulled along the x-axis by a student. The box slides across a rough surface, and its position x varies with time t according to the equation $x = 0.5t^3 + 2t$, where x is in meters and t is in seconds.

(a) Determine the speed of the box at time $t = 0$.

(b) Determine the following as functions of time t.

i. The kinetic energy of the box

ii. The net force acting on the box

iii. The power being delivered to the box

(c) Calculate the net work done on the box in the interval $t = 0$ to $t = 2$ s.

(d) Indicate below whether the work done on the box by the student in the interval $t = 0$ to $t = 2$ s would be greater than, less than, or equal to the answer in part (c).

____Greater than ____Less than ____Equal to

Justify your answer.

a) $x = 0.5t^3 + 2t$

$v = \frac{dx}{dt} = 1.5t^2 + 2 \xrightarrow{t=0} v(0) = \boxed{2\,\frac{m}{s}}$

b) i) $K = \frac{1}{2}mv^2 = \frac{1}{2}m(1.5t^2+2)^2 = \boxed{50(1.5t^2+2)^2}$

ii) $F_{NET} = ma = m\frac{dv}{dt} = m\frac{d}{dt}(1.5t^2+2) = m(3t) = \boxed{300t}$

iii) $P = F \cdot v = (300t)(1.5t^2+2) = \boxed{450t^3 + 600t}$

$$W = \int_{t=0}^{2s} P\,dt = \int_0^2 (450t^3 + 600t)\,dt = \left(\frac{450t^4}{4} + \frac{600t^2}{2}\right)\Big|_0^2$$

$$= \left(\frac{225t^4}{2} + 300t^2\right)\Big|_0^2 = \frac{225(2)^4}{2} + 300(2)^2 = 3000\text{ J}$$

d) Greater than

$W_{student} = W_{net} + W_{friction}$

5.1 - Momentum & Impulse

Wednesday, March 15, 2017 11:05 AM

Momentum

- Momentum is a vector describing how difficult it is to stop a moving object
- Total momentum is the sum of individual momenta
- $\vec{p} = m\vec{v}$
- Units are kg·m/s or N·s

Example 1: Changing Momentum

- An Aichi D3A bomber mass (3600 kg) departs from its aircraft carrier with a velocity of 85 m/s due east. What is its momentum?

$$\vec{p}_i = m\vec{v} = (3600\,kg)(85\,\tfrac{m}{s}) = \boxed{306{,}000\,\tfrac{kg\cdot m}{s}}$$

- After it drops its payload, its new mass is 3000 kg and it attains a cruising speed of 120 m/s . What is its new momentum?

$$\vec{p}_f = m\vec{v} = (3000\,kg)(120\,\tfrac{m}{s}) = \boxed{360{,}000\,\tfrac{kg\cdot m}{s}}$$

Impulse

- As you observed in the previous problem, momentum can change
- A change in momentum is known as an impulse (J)
- $\vec{J} = \Delta\vec{p}$

Example 2: Impulse

- The D3A bomber, which had a momentum of 3.6e5 kg·m/s, comes to halt on the ground. What impulse is applied?

$$\Delta p = p_f - p_i = 0 - 3.6\cdot 10^5\,\tfrac{kg\cdot m}{s} = \boxed{-3.6\cdot 10^5\,\tfrac{kg\cdot m}{s}}$$

Relationship Between Force and Impulse

- $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}$

Example 3: Force from Momentum

- The momentum of an object as a function of time is given by $p=kt^2$, where k is a constant. What is the equation for the force causing this motion?

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(kt^2) = k\frac{d}{dt}(t^2) = \boxed{2kt}$$

Impulse-Momentum Theorem

$$\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \int_0^t F \cdot dt = \int_{p_i}^{p_f} d\vec{p} \Rightarrow$$

$$\boxed{\vec{J} = \vec{F}\Delta t = \Delta\vec{p}}$$

Example 4: Impulse-Momentum

- A 6-kg block, sliding to the east across a horizontal, frictionless surface with a momentum of 30 kg·m/s, strikes an obstacle. The obstacle exerts an impulse of 10 N·s to the west on the block. Find the speed of the block after the collision.

$$\vec{J} = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i \Rightarrow m\vec{v}_f = \vec{J} + m\vec{v}_i \Rightarrow$$

$$\vec{v}_f = \frac{\vec{J} + m\vec{v}_i}{m} \Rightarrow \vec{v}_f = \frac{-10\,N\cdot s + 30\,\frac{kg\cdot m}{s}}{6\,kg} \Rightarrow$$

$$\boxed{\vec{v}_f = 3.33\,\tfrac{m}{s}\ \text{east}}$$

Example 5: Water Gun

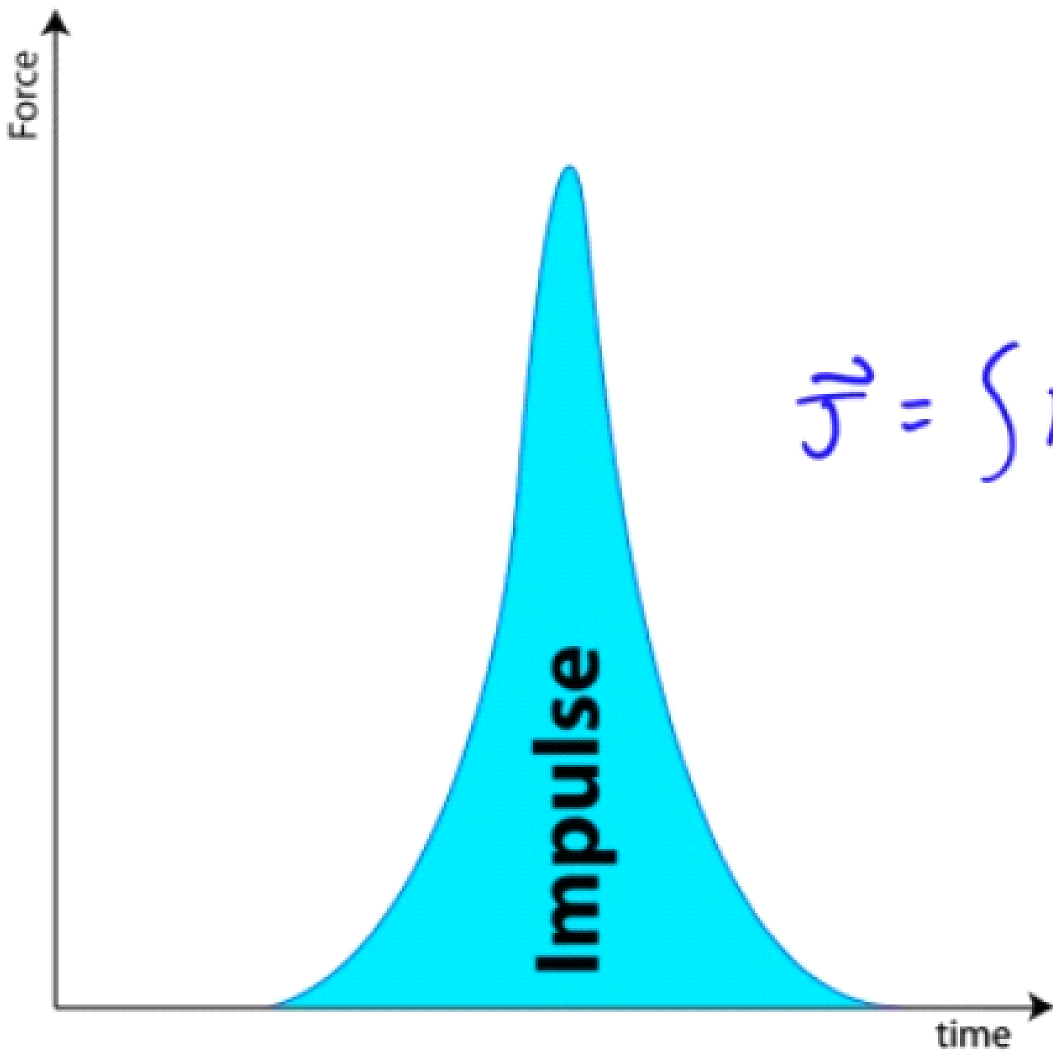
- A girl with a water gun shoots a stream of water than ejects 0.2 kg of water per second horizontally at a speed of 10 m/s. What horizontal force must the girl apply on the gun in order to hold it in position?

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{dm}{dt}\vec{v} = \left(0.2\frac{kg}{s}\right)\left(10\frac{m}{s}\right) \Rightarrow$$

$$\boxed{\vec{F} = 2N}$$

Impulse from F-t Graphs

- Impulse is the area under a Force-time graph
- Impulse is equivalent to a change in momentum



$$\vec{J} = \int \vec{F}\,dt$$

Example 6: Impulse from Force

- A force $F(t)=t^3$ is applied to a 10 kg mass. What is the total impulse applied to the object between 1 and 3 seconds?

$$F = \frac{dp}{dt} \Rightarrow dp = F dt \Rightarrow dp = t^3 dt \Rightarrow$$

$$\int_{p_i}^{p_f} dp = \int_{t=1}^{t=3s} t^3 dt \Rightarrow p_f - p_i = \frac{t^4}{4}\Big|_1^3 \Rightarrow$$

$$\Delta p = \frac{81}{4} - \frac{1}{4} = \frac{80}{4} \Rightarrow$$

$$\Delta p = \boxed{20 \frac{kg \cdot m}{s} = J}$$

5.2 - Conservation of Linear Momentum

Wednesday, March 15, 2017 11:05 AM

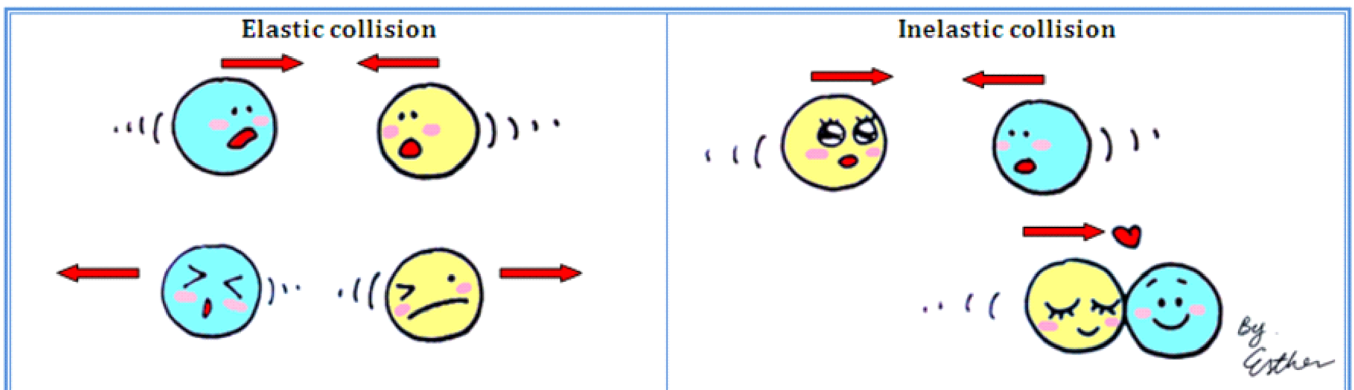
Collisions and Explosions

- In the case of a collision or explosion, if you add up the individual momentum vectors of all the objects before the event, you'll find that they are equal to the sum of momentum vectors of the objects after the event
- Written mathematically, the law of conservation of linear momentum states
- $\vec{p}_{initial} = \vec{p}_{final}$

Solving Momentum Problems

1. Identify all the objects in the system
2. Determine the momenta of the objects before the event. Use variables for any unknowns
3. Determine the momenta of the objects after the event. Use variables for any unknowns
4. Add up all the momenta from before the event and set equal to the momenta after the event
5. Solve for any unknowns

Types of Collisions



Objects that collide move **separately** after collision.	Objects that collide move **together** after collision.
Total momentum and **total energy** of the system are **conserved.**	
Kinetic energy is **conserved.**	**Kinetic energy** is **NOT conserved.**

- Elastic collision
  - Kinetic energy is conserved

- Inelastic collision
    - Kinetic energy is not conserved

Example 1: Traffic Collision

- A 2000-kg car traveling 20 m/s collides with a 1000-kg car at rest. If the 2000-kg car has a velocity of 6.67 m/s after the collision, find the velocity of the 1000-kg car after the collision

Objects	P_b $\left(\frac{kg\cdot m}{s}\right)$	P_a $\left(\frac{kg\cdot m}{s}\right)$
Car A	$2000\cdot 20 = 40{,}000$	$2000\cdot 6.67 = 13{,}340$
Car B	0	$1000\cdot V_B$
Total	$40{,}000$	$= 13{,}340 + 1000V_B$

$$40{,}000 - 13{,}340 = 13{,}340 + 1000V_B - 13{,}340$$

$$26{,}660 = 1000\,V_B$$

$$V_B = 26.7\ \text{m/s}$$

Example 2: Collision of Two Moving Objects

- On a snow-covered road, a car with a mass of 1100 kg collides head-on with a van having a mass of 2500 kg traveling at 8 m/s
- As a result of the collision, the vehicles lock together and immediately come to rest.
- Calculate the speed of the car immediately before the collision

Objects	P_b	P_a
Car	$1100 \cdot V_{car}$	$\emptyset$
Van	$2500 \cdot (-8) = -20{,}000$	
Total	$1100V_{car} - 20{,}000$	$= \emptyset$

$$1100V_{car} = 20{,}000$$

$$V_{car} = \boxed{18.2 \frac{m}{s}}$$

Example 3: Recoil Velocity

- A 4-kg rifle fires a 20-gram bullet with a velocity of 300 m/s. Find the recoil velocity of the rifle

Objects	P_b	P_a
Rifle	$\emptyset$	$4V_{recoil}$
Bullet		$(.02)(300) = 6$
Total	$\emptyset$	$= 4V_r + 6$

$$4V_r = -6$$

$$\boxed{V_r = -1.5 \frac{m}{s}}$$

Example 4: Atomic Collision

- A proton (mass=m) and a lithium nucleus (mass=7m) undergo an elastic collision as shown below.

- Find the velocity of the lithium nucleus following the collision

Objects	P_b	P_a
Proton	$1000m$	mV_p
Li Nuc	Ø	$7mV_L$
Total	$1000m = mV_p + 7mV_L$	

$1000 = V_p + 7V_L$

$V_p = 1000 - 7V_L$

$V_p = 1000 - 7(250 \, m/s)$

$\boxed{V_p = -750 \, m/s}$

$K_B = K_A$

$\frac{1}{2}m(1000)^2 = \frac{1}{2}m(1000 - 7V_L)^2 + \frac{1}{2}(7m)V_L^2$

$1000^2 = (1000 - 7V_L)^2 + 7V_L^2 \Rightarrow$

$10^6 = 10^6 - 14000V_L + 49V_L^2 + 7V_L^2 \Rightarrow$

$0 = -14000V_L + 56V_L^2$

$56V_L^2 = 14000V_L$

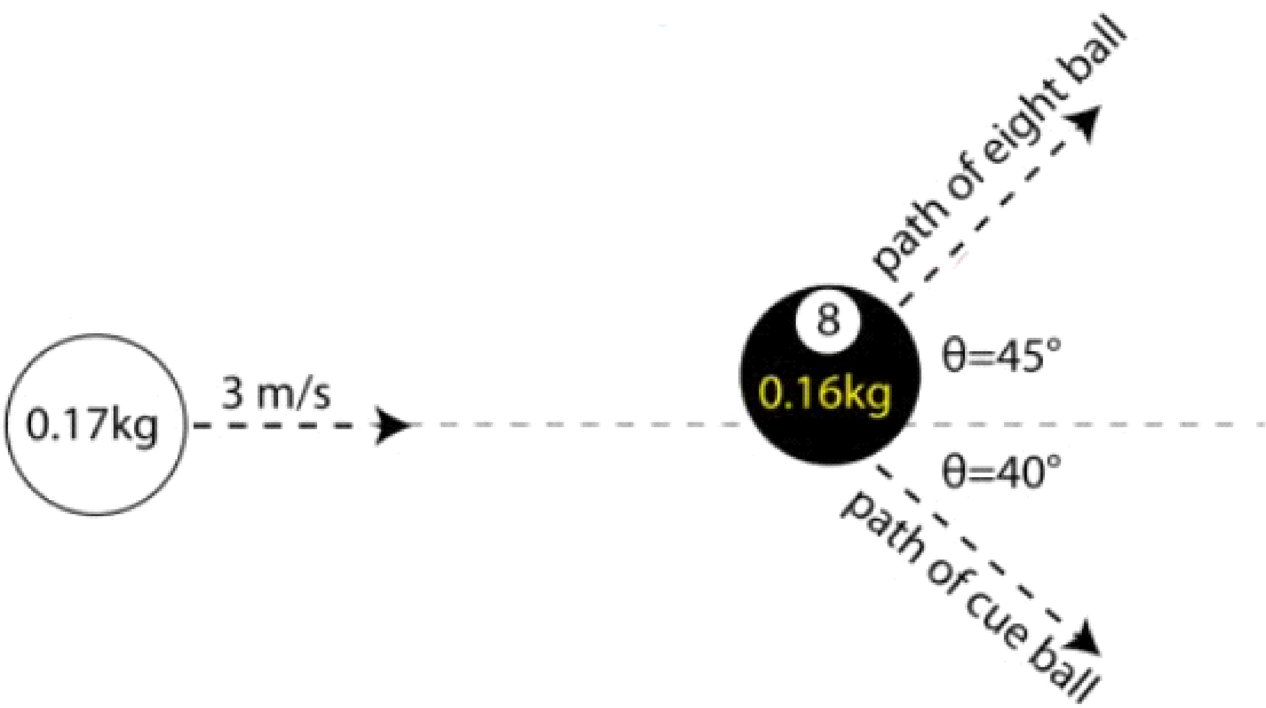
$56V_L = 14000$

$V_L = 250 \, m/s$

$V_L = 0$ or $\boxed{V_L = 250 \, m/s}$

Example 5: Collisions in Multiple Dimensions

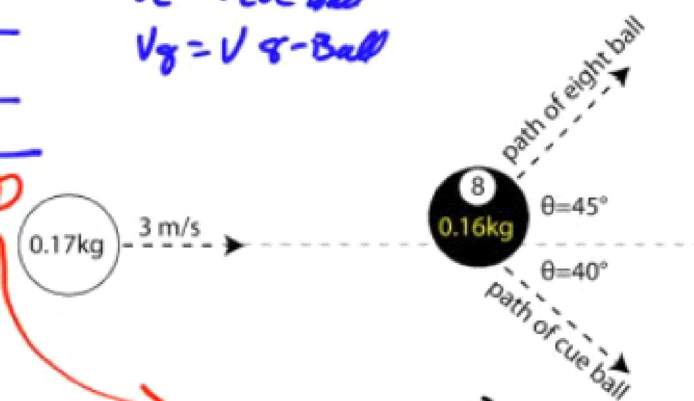
- Bert strikes a cur ball of mass 0.17 kg , giving it a velocity of 3 m/s in the x-direction. When the cue ball strikes the eight ball (mass=0.16kg), previously at rest, the eight ball is deflected 45 degrees from the cur ball's previous path, and the cue ball is deflected 40 degrees in the opposite direction. Find the velocity of the cue ball and the eight ball after the collision



$V_c = V_{cue\ ball}$

$V_8 = V_{8\text{-}Ball}$

Objects	P_{b_x}	P_{a_x}
Cue B	$.17(3) = .51$	$.17 V_c \cos 40°$
8 Ball	∅	$.16 V_8 \cos 45°$
Total	$.51$	$= .13 V_c + .113 V_8$ ①



Objects	P_{b_y}	P_{a_y}
Cue B	0	$.17 V_c \sin(-40°)$
8 Ball	0	$.16 V_8 \sin(45°)$
Total	0	$= -.109 V_c + .113 V_8$ ②

$$.109 V_c = .113 V_8$$

$$V_c = 1.04 V_8$$

$$.51 = .13(1.04 V_8) + .113 V_8$$

$$.51 = .248 V_8$$

$$\boxed{V_8 = 2.06 \text{ m/s}}$$

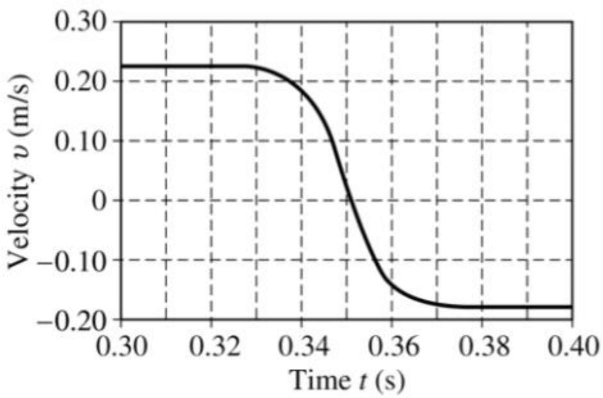
$$V_c = 1.04 (2.06 \text{ m/s})$$

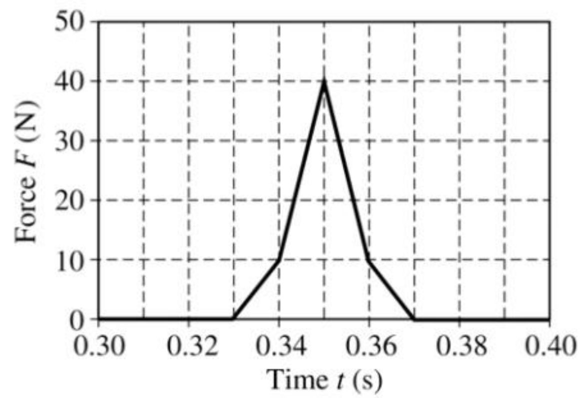
$$\boxed{V_c = 2.14 \text{ m/s}}$$

2001 Free Response Question 1

Mech 1.

A motion sensor and a force sensor record the motion of a cart along a track, as shown above. The cart is given a push so that it moves toward the force sensor and then collides with it. The two sensors record the values shown in the following graphs.





(a) Determine the cart's average acceleration between $t = 0.33$ s and $t = 0.37$ s.

(b) Determine the magnitude of the change in the cart's momentum during the collision.

(c) Determine the mass of the cart.

(d) Determine the energy lost in the collision between the force sensor and the cart.

a) $a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t} = \frac{-.18\,\frac{m}{s} - .22\,\frac{m}{s}}{.04\,s} = \frac{-.4\,\frac{m}{s}}{.04\,s} = \boxed{-10\,m/s^2}$

b) $\Delta p = \int_{t_1}^{t_2} \vec{F}\cdot dt = \text{Area} = \frac{1}{2}(.01)(10) + (.02)(10) + \frac{1}{2}(.02)(30) + \frac{1}{2}(.01)(10) = \boxed{0.6\,\frac{kg\cdot m}{s}}$

c) $\Delta p = m\Delta v \Rightarrow m = \frac{\Delta p}{\Delta v} = \frac{0.6\,\frac{kg\cdot m}{s}}{0.4\,\frac{m}{s}} = \boxed{1.5\,kg}$

d) $\Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \frac{m}{2}(v_f^2 - v_i^2) = \frac{1.5}{2}(.22^2 - .18^2) \Rightarrow$

$$\boxed{\Delta K = 0.012\,J}$$

2002 Free Response Question 1

A crash test car of mass 1,000 kg moving at constant speed of 12 m/s collides completely inelastically with an object of mass M at time $t = 0$. The object was initially at rest. The speed v in m/s of the car-object system after the collision is given as a function of time t in seconds by the expression

$$v = \frac{8}{1 + 5t}.$$

(a) Calculate the mass M of the object.

(b) Assuming an initial position of $x = 0$, determine an expression for the position of the car-object system after the collision as a function of time t.

(c) Determine an expression for the resisting force on the car-object system after the collision as a function of time t.

(d) Determine the impulse delivered to the car-object system from $t = 0$ to $t = 2.0$ s.

a) $P_b = P_a \Rightarrow (1000)(12) + M(\emptyset) = (1000+M)\cdot v(t=0) \Rightarrow$

$12000 = (1000+M)(8) \Rightarrow 12000 = 8000 + 8M \Rightarrow$

$8M = 4000 \Rightarrow \boxed{M = 500 \text{ kg}}$

b) $v(t) = \frac{8}{1+5t} \Rightarrow x(t) = \int_0^t \frac{8\,dt}{1+5t} = \frac{8}{5}\int_0^t \frac{5\,dt}{1+5t} \xrightarrow[du = 5dt]{u = 1+5t}$

$x(t) = \frac{8}{5}\ln(1+5t)\Big|_0^t = \frac{8}{5}\left[\ln(1+5t) - \ln(1)\right] \Rightarrow$

$x(t) = \frac{8}{5}\ln(1+5t)$

c) $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{8}{1+5t}\right) = 8\frac{d}{dt}\left((1+5t)^{-1}\right) = 8\left[-(1+5t)^{-2}5\right] =$

$\frac{-40}{(1+5t)^2}$

$F = ma = 1500\left(\frac{-40}{(1+5t)^2}\right) = \boxed{\frac{-60{,}000}{(1+5t)^2}}$

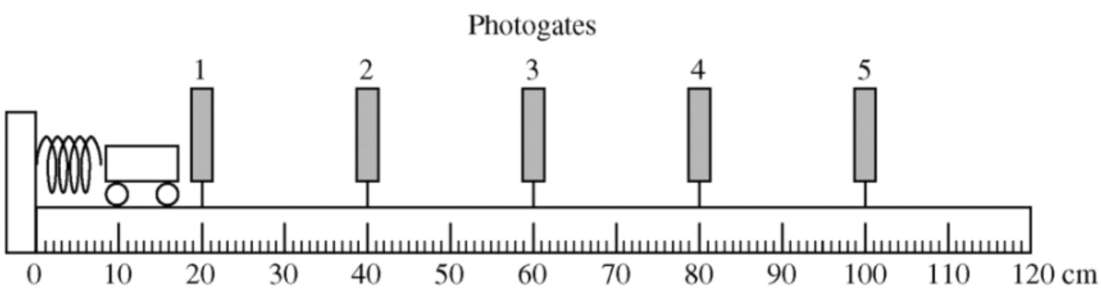
d) $J = \int F\,dt = \int_{t=0}^{2s} \frac{-60000}{(1+5t)^2}\,dt = \frac{-60000}{5}\int_0^2 (5t+1)^{-2}\,5\,dt$

$= \frac{-60,000}{5}\left[\frac{(1+5t)^{-1}}{-1}\right]\Big|_0^2 = -12,000\left[\frac{-1}{1+5t}\right]\Big|_0^2 \Rightarrow$

$J = -12,000\left[-\frac{1}{11} - -1\right] = -12,000\left[-\frac{1}{11}+1\right] \Rightarrow$

$\boxed{J = -10,909\ \frac{kg\cdot m}{s}}$

2014 Free Response Question 1



Mech. 1.

In an experiment, a student wishes to use a spring to accelerate a cart along a horizontal, level track. The spring is attached to the left end of the track, as shown in the figure above, and produces a nonlinear restoring force of magnitude $F_s = As^2 + Bs$, where s is the distance the spring is compressed, in meters. A measuring tape, marked in centimeters, is attached to the side of the track. The student places five photogates on the track at the locations shown.

(a) Derive an expression for the potential energy U as a function of the compression s. Express your answer in terms of A, B, s, and fundamental constants, as appropriate.

In a preliminary experiment, the student pushes the cart of mass 0.30 kg into the spring, compressing the spring 0.040 m. For this spring, $A = 200\ \mathrm{N/m^2}$ and $B = 150\ \mathrm{N/m}$. The cart is released from rest. Assume friction and air resistance are negligible only during the short time interval when the spring is accelerating the cart.

(b) Calculate the following:

i. The speed of the cart immediately after it loses contact with the spring

ii. The impulse given to the cart by the spring

In a second experiment, the student collects data using the photogates. Each photogate measures the speed of the cart as it passes through the gate. The student calculates a spring compression that should give the cart a speed of $0.320\ \mathrm{m/s}$ after the cart loses contact with the spring. The student runs the experiment by pushing the cart into the spring, compressing the spring the calculated distance, and releasing the cart. The speeds are measured with a precision of $\pm 0.002\ \mathrm{m/s}$. The positions are measured with a precision of ± 0.005 m.

Photogate	1	2	3	4	5
Cart speed (m/s)	0.412	0.407	0.399	0.374	0.338
Photogate position (m)	0.20	0.40	0.60	0.80	1.00

(c) On the axes below, plot the data points for the speed v of the cart as a function of position x. Clearly scale and label all axes, as appropriate.

(d)

i. Compare the speed of the cart measured by photogate 1 to the predicted value of the speed of the cart just after it loses contact with the spring. List a physical source of error that could account for the difference.

ii. From the measured speed values of the cart as it rolls down the track, give a physical explanation for any trend you observe.

a) $\vec{F} = -\frac{dU}{dl} \Rightarrow dU = -F\,dl \Rightarrow U = -\int F\,dl \Rightarrow$

$$U = -\int_{s=0}^{s} (As^2 + Bs)\,ds = -\left(\frac{As^3}{3} + \frac{Bs^2}{2}\right)\Big|_0^s = -\frac{As^3}{3} - \frac{Bs^2}{2}$$

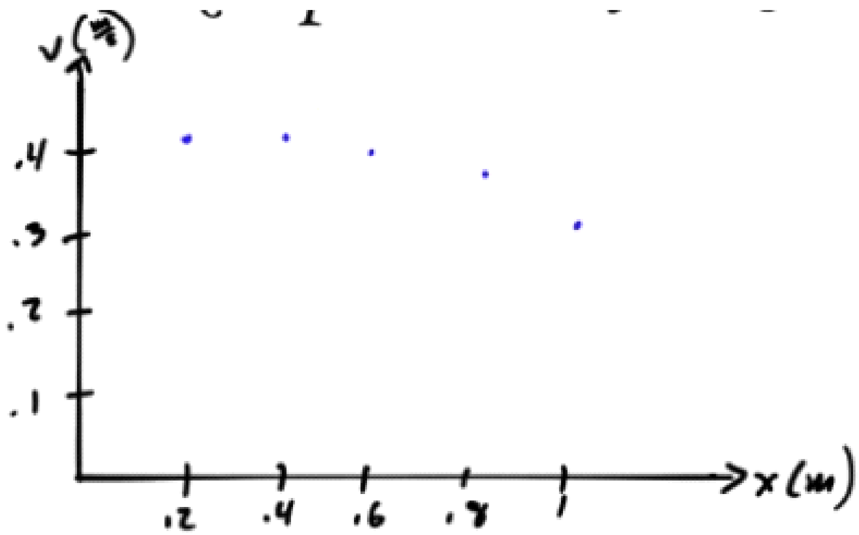
$$\boxed{U_s = \frac{As^3}{3} + \frac{Bs^2}{2}}$$

b) $E_i = E_f \Rightarrow U_{s_i} = K_f \Rightarrow \frac{As^3}{3} + \frac{Bs^2}{2} = \frac{1}{2}mv^2 \Rightarrow \frac{(200)(.04)^3}{3} + \frac{(150)(.04)^2}{2} =$

$\frac{1}{2}(.3)v^2 \Rightarrow .1243 = \frac{1}{2}(.3)v^2 \Rightarrow v^2 = .828 \Rightarrow \boxed{v = 0.91\ \frac{m}{s}}$

bii) $J = \Delta p = mv_f - mv_i = 0.3(.91 - 0) = 0.273\ \frac{kg \cdot m}{s}$

c)



5.3 - Center of Mass

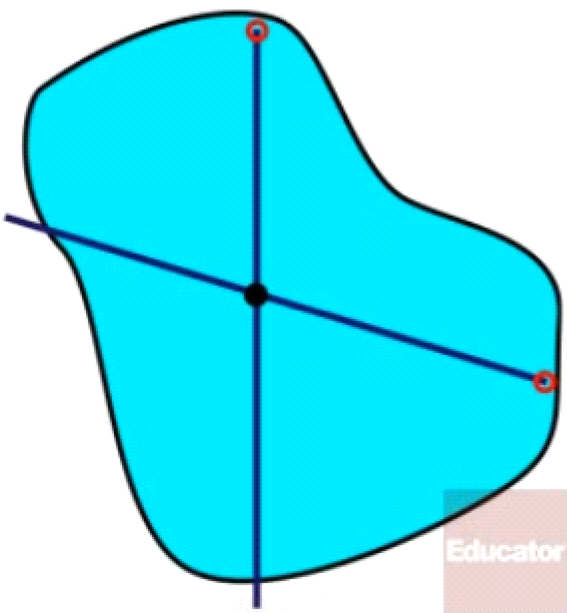
Wednesday, March 15, 2017 11:05 AM

Center of Mass

- Real objects are more complex than theoretical particles
- Treat entire object as if its entire mass were contained at a single point known as the object's center of mass (CM)
- Center of mass is the weighted average of the location of mass in an object

Find CM by Inspection

- For uniform density objects, CM is the geometric center
- For objects with multiple parts, find CM of each part and treat as a point
- For irregular objects, suspend object from two or more points and drop a plumb line. The line intersects at the center of mass



Calculating CM for Systems of Particles

- $r_{CM} = \frac{\sum m\vec{r}}{\sum m}$
- $x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{m_1 + m_2 + \cdots}$
- $y_{CM} = \frac{m_1 y_1 + m_2 y_2 + \cdots}{m_1 + m_2 + \cdots}$

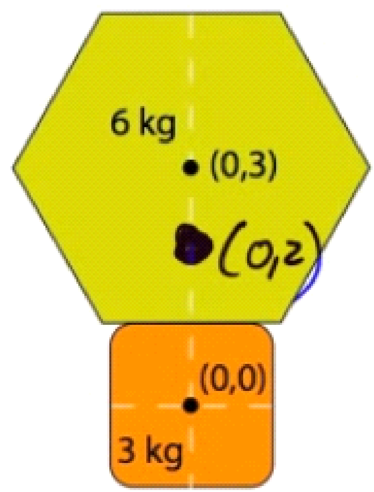
Example 1: Center of Mass (1D)

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{M} = \frac{(2kg)(2) + (6kg)(8m)}{8kg} = 6.5m$$

Example 2: CM of Continuous System

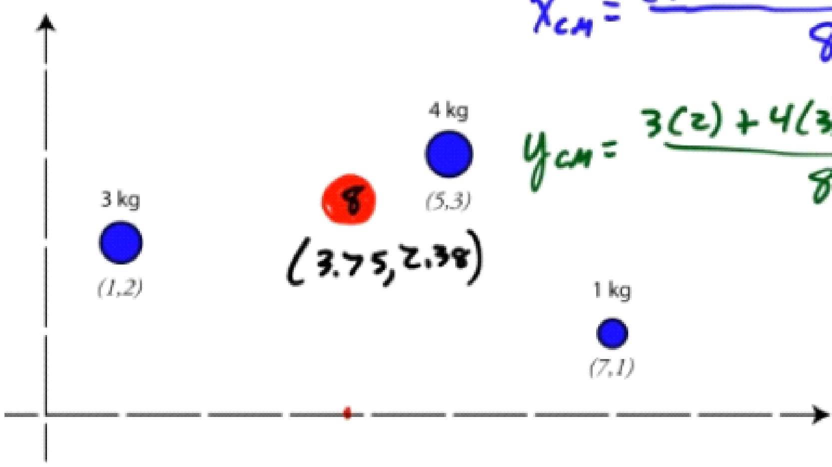
$$y_{cm} = \frac{m_1y_1 + m_2y_2}{M}$$

$$= \frac{(3kg)(0) + (6kg)(3)}{9kg} \Rightarrow y_{cm} = 2$$



(0,2)

Example 3: Center of Mass (2D)



$$x_{cm} = \frac{(3)(1) + 4(5) + 1(7)}{8} = 3.75$$

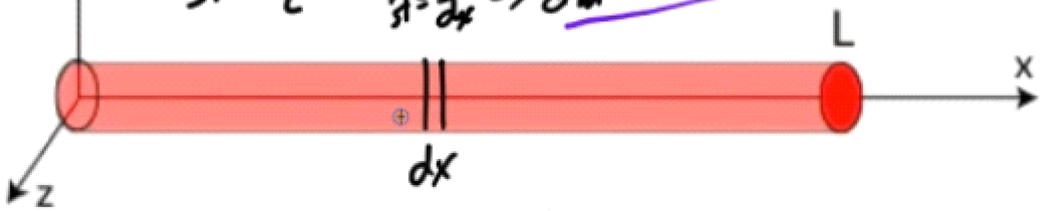
$$y_{cm} = \frac{3(2) + 4(3) + 1(1)}{8} = 2.38$$

Finding CM by Integration

- For more complex objects, you can find the center of mass by summing up all the little pieces of position vectors multiplied by the differential of mass and dividing by the total mass
- $\vec{r}_{CM} = \frac{\int \vec{r}\,dm}{M}$

Example 4: CM of a Uniform Rod

$$\lambda = \frac{M}{L} \qquad \lambda = \frac{dM}{dx} \Rightarrow dm = \lambda\,dx$$



$$\vec{r}_{cm} = \frac{\int \vec{r}\,dm}{M} \Rightarrow \vec{r}_{cm} = \frac{\int x\,dm}{M} = \frac{\int x\,\lambda\,dx}{M} = \frac{\lambda}{M}\int_0^L x\,dx \Rightarrow$$

$$\vec{r}_{cm} = \frac{\lambda}{M}\,\frac{x^2}{2}\Big|_0^L = \frac{\lambda}{M}\,\frac{L^2}{2} \xrightarrow{M = L\lambda} \frac{\cancel{\lambda}L^{\cancel{2}}}{\cancel{L}\cancel{\lambda}2} = \boxed{\frac{L}{2}}$$

Example 5: CM of a Non-Uniform Rod

- Find the center of mass of a non-uniform rod of length L and mass M whose density is given by λ=kx

$$M = \int_{x=0}^{L} \lambda\, dx = \int_0^L kx\, dx = \frac{kL^2}{2}$$

$$\vec{r}_{CM} = \frac{\int \vec{r}\, dm}{M} \xrightarrow[\substack{\lambda = kx \\ dm = kx\, dx}]{\vec{r} = x} \vec{r}_{CM} = \frac{\int x\, kx\, dx}{M} \Rightarrow$$

$$\vec{r}_{CM} = \frac{k}{M} \int_0^L x^2\, dx = \frac{k}{M} \frac{x^3}{3}\Big|_0^L = \frac{kL^3}{3M} \xrightarrow{M = \frac{kL^2}{2}}$$

$$\vec{r}_{CM} = \frac{\cancel{k} L^3\, 2}{3\, \cancel{k} \cancel{L^2}} = \boxed{\frac{2}{3} L}$$

Center of Mass Relationships

$$\vec{r}_{CM} = \frac{1}{M} \sum_i m_i \vec{r}_i \Rightarrow \vec{v}_{CM} = \frac{1}{M} \sum_i m_i \vec{v}_i \xrightarrow{\vec{p} = m\vec{v}}$$

$$\vec{v}_{CM} = \frac{1}{M} \sum_i \vec{p}_i \Rightarrow \vec{P}_{TOTAL} = M\vec{v}_{CM}$$

$$\vec{P}_{TOTAL} = M\vec{v}_{CM} \xrightarrow{\vec{F} = \frac{d\vec{p}}{dt}} \frac{d\vec{P}_{TOTAL}}{dt} = \vec{F}_{TOTAL} = \frac{d}{dt}(M\vec{v}_{CM}) \Rightarrow$$

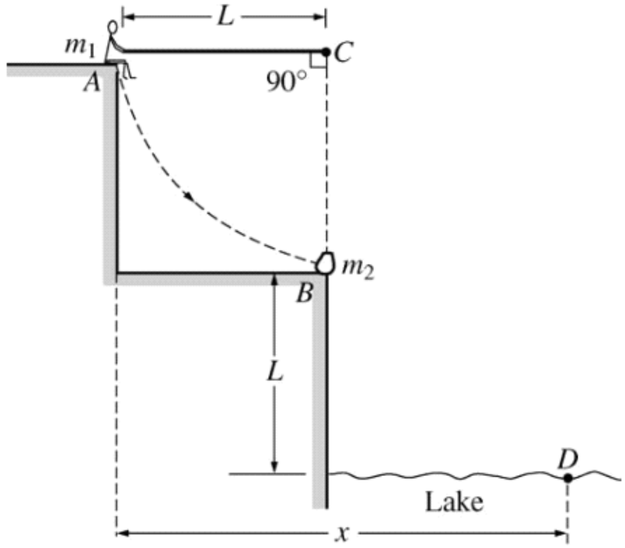
$$\vec{F}_{TOTAL} = M\vec{a}_{CM} \qquad \text{N2 Law}$$

Center of Gravity

- Center of Gravity refers to the location at which the force of gravity acts upon an object as if it were a point particle with all its mass focused at that point

- In a uniform gravitational field, Center of Gravity and Center of Mass are the same
- In a non-uniform gravitational field, they may be different

2004 Free Response Question 1



Mech. 1.

A rope of length L is attached to a support at point C. A person of mass m_1 sits on a ledge at position A holding the other end of the rope so that it is horizontal and taut, as shown above. The person then drops off the ledge and swings down on the rope toward position B on a lower ledge where an object of mass m_2 is at rest. At position B the person grabs hold of the object and simultaneously lets go of the rope. The person and object then land together in the lake at point D, which is a vertical distance L below position B. Air resistance and the mass of the rope are negligible. Derive expressions for each of the following in terms of m_1, m_2, L, and g.

(a) The speed of the person just before the collision with the object

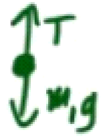
(b) The tension in the rope just before the collision with the object

(c) The speed of the person and object just after the collision

(d) The ratio of the kinetic energy of the person-object system before the collision to the kinetic energy after the collision

(e) The total horizontal displacement x of the person from position A until the person and object land in the water at point D.

a) $U_{g_A} = K_B \Rightarrow mgL = \frac{1}{2} m_1 v^2 \Rightarrow \underbrace{v^2 = 2gL} \Rightarrow \boxed{v = \sqrt{2gL}}$

b) 

$F_{NET_c} = T - m_1 g = m a_c = \frac{m_1 v^2}{L} \Rightarrow T = \frac{m_1 v^2}{L} + m_1 g \xrightarrow{v^2 = 2gL}$

$$T = \frac{m_1 2g \cancel{L}}{\cancel{L}} + m_1 g = \boxed{3 m_1 g}$$

c) $p_i = p_f \Rightarrow m_1 v_i = (m_1 + m_2) v_f \Rightarrow v_f = \left(\frac{m_1}{m_1 + m_2}\right) v_i = \boxed{\left(\frac{m_1}{m_1 + m_2}\right) \sqrt{2gL}}$

d)

$$K_b = \frac{1}{2} m_1 v_b^2 = \frac{1}{2} m_1 (2gL) = m_1 g L$$

$$K_a = \frac{1}{2}(m_1 + m_2) v_a^2 = \frac{1}{2}(m_1 + m_2)\left(\left(\frac{m_1}{m_1 + m_2}\right)\sqrt{2gL}\right)^2 \Rightarrow$$

$$K_a = \frac{1}{2}(m_1 + m_2)\frac{m_1^2}{(m_1 + m_2)^2}(2gL) = \left(\frac{m_1^2}{m_1 + m_2}\right) gL$$

$$\frac{K_b}{K_a} = \frac{\cancel{m_1} g \cancel{L}\,(m_1 + m_2)}{m_1^{\cancel{2}} g \cancel{L}} = \boxed{\frac{m_1 + m_2}{m_1}}$$

e) Vert

$\downarrow +y$

$v_i = 0$

$\Delta y = L$

$a = g$

$$\Delta y = \cancel{v_0 t}^{\,0} + \frac{1}{2} a_y t^2$$

$$L = \frac{1}{2} g t^2 \Rightarrow t = \sqrt{\frac{2L}{g}}$$

HORZ

$$\Delta x_{BD} = v_x t = \left(\left(\frac{m_1}{m_1 + m_2} \right) \sqrt{2gL} \right) \left(\sqrt{\frac{2L}{g}} \right) = \boxed{\frac{2Lm_1}{m_1 + m_2}}$$

$$X_{Tot} = \boxed{L + \frac{2Lm_1}{m_1 + m_2}} = \frac{L(m_1 + m_2)}{m_1 + m_2} + \frac{2Lm_1}{m_1 + m_2} = \frac{L(m_1 + m_2 + 2m_1)}{m_1 + m_2}$$

$$= \boxed{\frac{L(3m_1 + m_2)}{m_1 + m_2}}$$

6.1 - Uniform Circular Motion

Wednesday, March 15, 2017 11:06 AM

Uniform Circular Motion

- Object travels in a circular path at constant speed
- Distance around the circle is its circumference
  - $C = 2\pi r = \pi d$
- Average speed formula from kinematics still applies
  - $\bar{v} = \frac{d}{t} = 2\pi r$

Frequency

- Frequency is the number of revolutions or cycles which occur each second
- Symbol is f
- Units are 1/s, or Hertz (Hz)
- $f = \frac{number\ of\ cycles}{second} = \frac{number\ of\ revolution}{second}$

Period

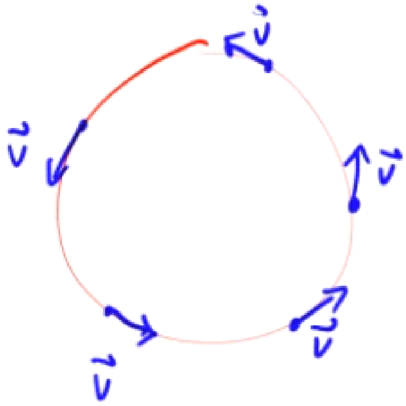
- Period is the time it takes for one complete revolution, or cycle.
- Symbol is T
- Unites are seconds (s)
- T = time for 1 cycle = time for 1 revolution

Frequency and Period

- $f = \frac{1}{T}$
- $T = \frac{1}{f}$

Centripetal Acceleration

- Is an object undergoing UCM accelerating?

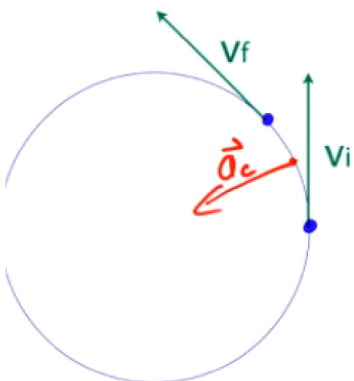


constant speed

yes, it is accelerating!

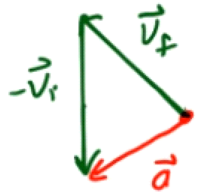
$\vec{a} = \frac{\Delta \vec{v}}{t} = \frac{\vec{v}_f - \vec{v}_i}{t}$

$\vec{v}_f - \vec{v}_i = \vec{v}_f + -\vec{v}_i$



centripetal acceleration

a_c



center-seeking

- Magnitude of Centripetal Acceleration
  - $a_c = \frac{v^2}{r}$

Centripetal Force

- If an object is traveling in a circle it is accelerating toward the center of the circle
- For an object to accelerate, there must be a net force
- We call this force a centripetal force (F_c)
- $F_c = \frac{mv^2}{r}$

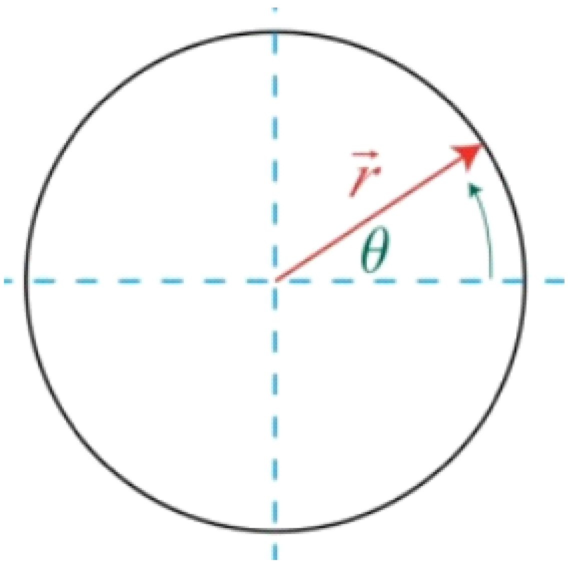
7.1 - Rotational Kinematics

Wednesday, March 15, 2017 8:21 PM

Radians and Degrees

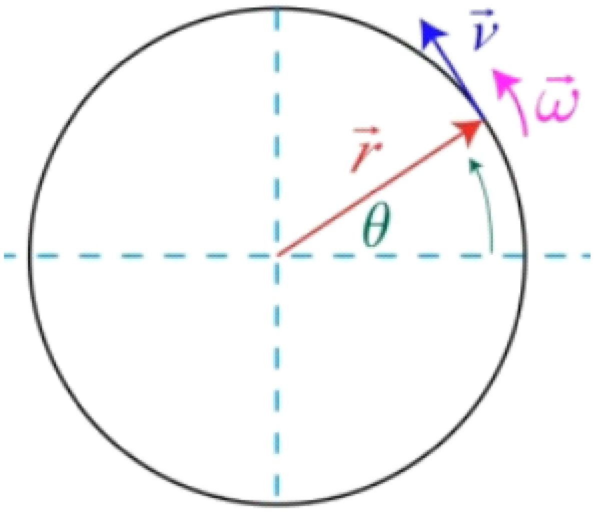
- In degrees, once around a circle is 360°
- In radians, once around a circle is 2π
- A radian measures a distance around an arc equal to the length of the arc's radius
- $\Delta s = C = 2\pi r$

Linear vs. Angular Displacement



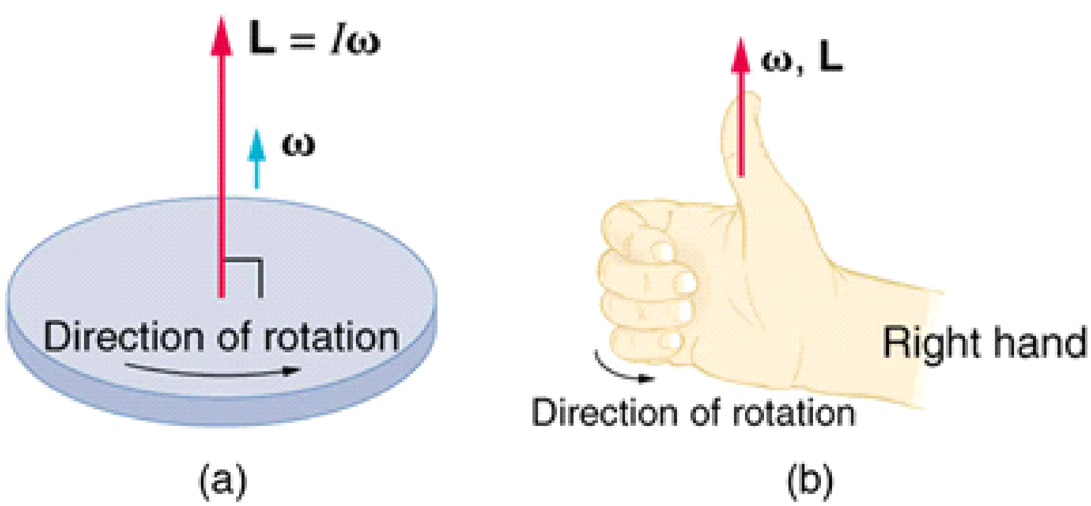
- Linear position / displacement given by Δr or Δs
- Angular position / displacement given by $\Delta\theta$
- $s = r\theta$
- $\Delta s = r\Delta\theta$

Linear vs. Angular Velocity



- Linear speed / velocity given by $\vec{v}$
- Angular speed / velocity given by $\vec{\omega}$
- $\vec{v} = \frac{d\vec{s}}{dt}$
- $\vec{\omega} = \frac{d\vec{\theta}}{dt}$

Direction of Angular Velocity



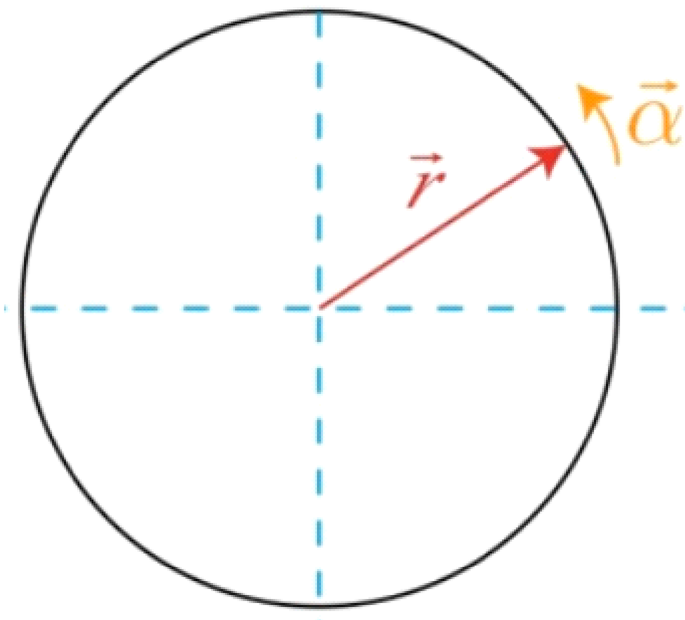
Converting Linear to Angular Velocity

$$v = \frac{ds}{dt} \xrightarrow{s = r\theta} \frac{d r\theta}{dt} \xrightarrow{r \text{ constant}}$$

$$v = r\frac{d\theta}{dt} \xrightarrow{\omega = \frac{d\theta}{dt}}$$

$$\vec{v} = r\vec{\omega}$$

Linear vs. Angular Acceleration



- Linear acceleration is given by $\vec{a}$
- Angular acceleration is given by $\vec{\alpha}$
- $\vec{a} = \frac{d\vec{v}}{dt}$
- $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$

Kinematic Variable Parallels

Variable	Translational	Angular
Displacement	Δs	$\Delta\theta$
Velocity	v	ω
Acceleration	a	α

Time	t	t

Variable Translations

Variable	Translational	Angular
Displacement	$s = r\theta$	$\theta = \frac{s}{r}$
Velocity	$v = r\omega$	$\omega = \frac{v}{r}$
Acceleration	$a = r\alpha$	$\alpha = \frac{a}{r}$
Time	$t = t$	$t = t$

Kinematic Equation Parallels

Rotational Motion (α = constant)	Linear Motion (a = constant)
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \frac{1}{2}(\omega_0 + \omega)t$	$x = \frac{1}{2}(v_0 + v)t$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$

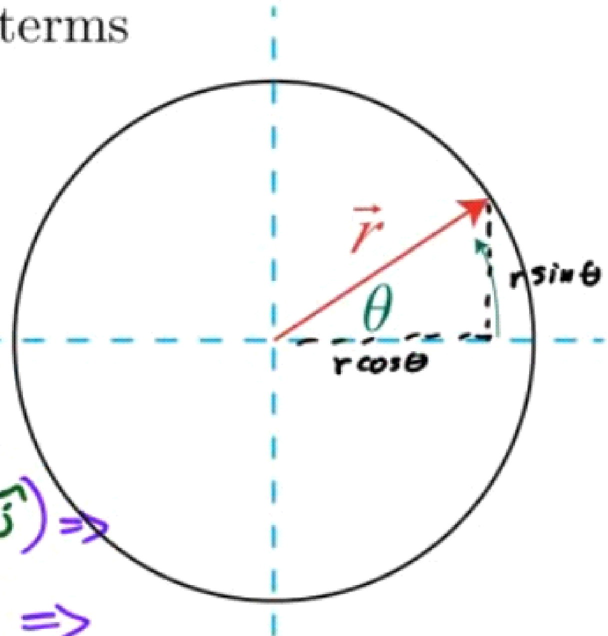
Centripetal Acceleration

- Express position vector in terms of unit vectors.

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} \xrightarrow[y(t)=r\sin\theta]{x(t)=r\cos\theta}$$

$$\vec{r}(t) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} \xrightarrow{\theta=\omega t}$$

$$\vec{r}(t) = r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j}$$



$$v = \frac{dr}{dt} = \frac{d}{dt}\left(r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j}\right) \Rightarrow$$

$$v = -\omega r\sin(\omega t)\hat{i} + \omega r\cos(\omega t)\hat{j} \Rightarrow$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(-\omega r\sin(\omega t)\hat{i} + \omega r\cos(\omega t)\hat{j}\right) \Rightarrow$$

$$a = -\omega^2 r\cos(\omega t)\hat{i} - \omega^2 r\sin(\omega t)\hat{j} \Rightarrow$$

$$a = -\omega^2\left(r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j}\right) \xrightarrow{r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j} = \vec{r}}$$

$$a = -\omega^2 r \qquad |a| = \omega^2 r \xrightarrow[\omega=\frac{v}{r}]{v=\omega r} |a| = \frac{v^2}{r^2}r = \frac{v^2}{r}$$

Example: Wheel in Motion

- A wheel of radius r and mass M undergoes a constant angular acceleration of magnitude α.
- What is the speed of the wheel after it has completed on complete turn, assuming it started from rest?

$$\omega_f^2 = \cancel{\omega_0^2}^{\to 0} + 2\alpha\Delta\theta$$

$$\omega_f^2 = 2\alpha(2\pi) = 4\pi\alpha$$

$$\omega_f = \sqrt{4\pi\alpha}$$

$$v = r\omega = \boxed{r\sqrt{4\pi\alpha}}$$

2003 Free Response Question 3

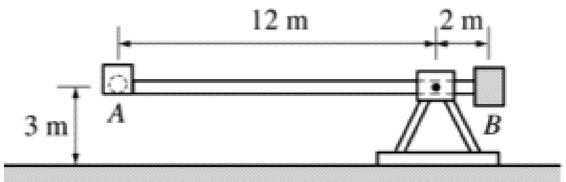


Figure 1

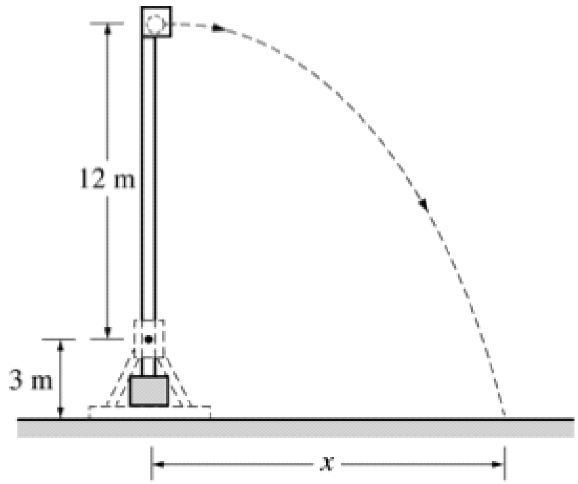


Figure 2

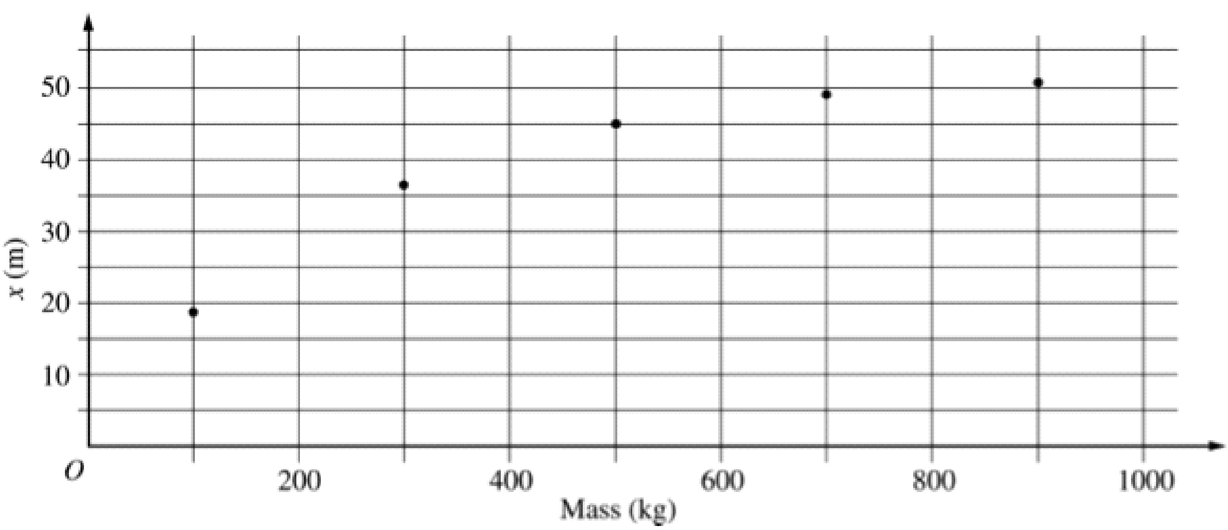
Mech. 3.

Some physics students build a catapult, as shown above. The supporting platform is fixed firmly to the ground. The projectile, of mass 10 kg, is placed in cup A at one end of the rotating arm. A counterweight bucket B that is to be loaded with various masses greater than 10 kg is located at the other end of the arm. The arm is released from the horizontal position, shown in Figure 1, and begins rotating. There is a mechanism (not shown) that stops the arm in the vertical position, allowing the projectile to be launched with a horizontal velocity as shown in Figure 2.

(a) The students load five different masses in the counterweight bucket, release the catapult, and measure the resulting distance x traveled by the 10 kg projectile, recording the following data.

Mass (kg)	100	300	500	700	900
x (m)	18	37	45	48	51

i. The data are plotted on the axes below. Sketch a best-fit curve for these data points.



ii. Using your best-fit curve, determine the distance x traveled by the projectile if 250 kg is placed in the counterweight bucket.

(b) The students assume that the mass of the rotating arm, the cup, and the counterweight bucket can be neglected. With this assumption, they develop a theoretical model for x as a function of the counterweight mass using the relationship $x = v_x t$, where v_x is the horizontal velocity of the projectile as it leaves the cup and t is the time after launch.

i. How many seconds after leaving the cup will the projectile strike the ground?

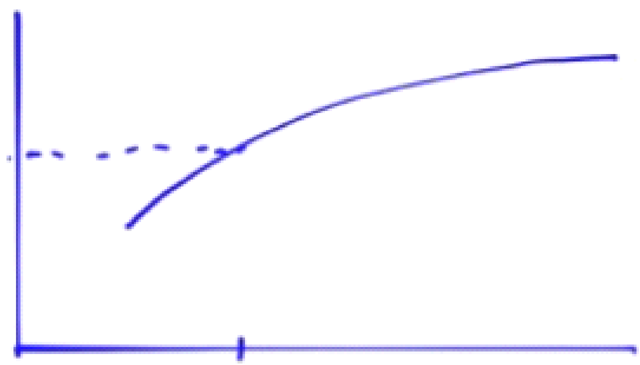
ii. Derive the equation that describes the gravitational potential energy of the system relative to the ground when in the position shown in Figure 1, assuming the mass in the counterweight bucket is M.

iii. Derive the equation for the velocity of the projectile as it leaves the cup, as shown in Figure 2.

(c)

i. Complete the theoretical model by writing the relationship for x as a function of the counterweight mass using the results from (b)i and (b)iii.

ii. Compare the experimental and theoretical values of x for a counterweight bucket mass of 300 kg. Offer a reason for any difference.



$x \approx 33\text{m}$

b i) VERT

$v_i = 0$

v_f

↓ +y $\Delta y = 15\text{m}$

$a_y = 10\,\text{m/s}^2$

$t =$

$$\Delta y = \cancel{v_i t}^{\to 0} + \tfrac{1}{2} a_y t^2$$

$$t = \sqrt{\frac{2\Delta y}{a}} = \sqrt{\frac{2(15\text{m})}{10}} = \boxed{1.73\,\text{s}}$$

b ii)

$$U_i = U_B + U_P = m_B g h_B + m_P g h_P = (10)(3)(M+10) = 300 + 30M$$

iii) $U_f + K_f = U_i \Rightarrow U_f = 1 \cdot 10 \cdot M + 15(10)(10) = 1500 + 10M$

$$K_f = \tfrac{1}{2}(10)(v_x^2) + \tfrac{1}{2} M v_B^2$$

$$U_i = 300 + 30M$$

$$300 + 30M = 10M + 1500 + 5v_x^2 + \tfrac{1}{2} M v_B^2$$

$$\omega_B = \omega_A \xrightarrow[\omega = \frac{v}{r}]{v = \omega r} \frac{v_b}{2} = \frac{v_x}{12} \Rightarrow v_B = \frac{v_x}{6}$$

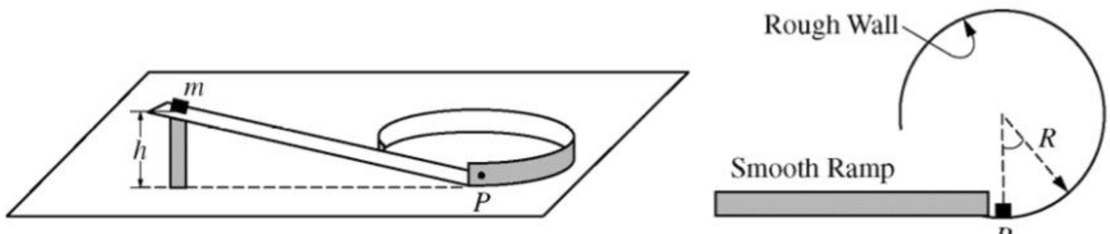
$$300 + 30M = 10M + 1500 + 5v_x^2 + \tfrac{1}{2} M\left(\frac{v_x^2}{36}\right) \Rightarrow 20M = 1200 + 5v_x^2$$

$$+ \frac{M v_x^2}{72} \Rightarrow 20M - 1200 = v_x^2\left(5 + \frac{M}{72}\right) \Rightarrow \boxed{v_x = \sqrt{\frac{20M - 1200}{5 + M/72}}}$$

c) $x = v_x t = 1.73\sqrt{\dfrac{20M - 1200}{5 + M/72}}$

ii) $\left.\begin{aligned} x_{theo}(M = 300) &= 39.6\,m \\ x_{act}(M = 300) &= 37\,m \end{aligned}\right\}$?

2014 Free Response Question 2



Side View

Top View

Mech. 2.

A small block of mass m starts from rest at the top of a frictionless ramp, which is at a height h above a horizontal tabletop, as shown in the side view above. The block slides down the smooth ramp and reaches point P with a speed v_0 . After the block reaches point P at the bottom of the ramp, it slides on the tabletop guided by a circular vertical wall with radius R, as shown in the top view. The tabletop has negligible friction, and the coefficient of kinetic friction between the block and the circular wall is μ .

(a) Derive an expression for the height of the ramp h. Express your answer in terms of v_0 , m, and fundamental constants, as appropriate.

A short time after passing point P, the block is in contact with the wall and moves with a speed of v .

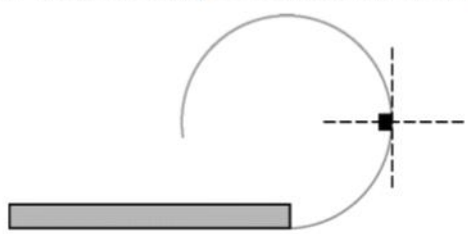
(b)

i. Is the vertical component of the net force on the block upward, downward, or zero?

____ Upward ____ Downward ____ Zero

Justify your answer.

ii. On the figure below, draw an arrow starting on the block to indicate the direction of the horizontal component of the net force on the moving block when it is at the position shown.



Top View

Justify your answer.

Express your answers to the following in terms of v_0 , v , m, R, μ , and fundamental constants, as appropriate.

(c) Determine an expression for the magnitude of the normal force N exerted on the block by the circular wall as a function of v .

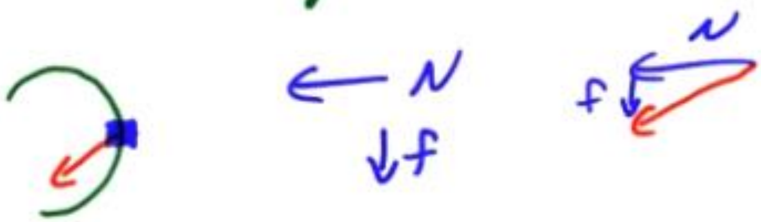
(d) Derive an expression for the magnitude of the tangential acceleration of the block at the instant the block has attained a speed of v .

(e) Derive an expression for $v(t)$, the speed of the block as a function of time t after passing point P on the track.

a) $K_f = U_{g_i} \Rightarrow \frac{1}{2} m v_0^2 = mgh \Rightarrow \boxed{h = \frac{v_0^2}{2g}}$

b) i) $a_y = 0 \Rightarrow F_{NET_y} = 0$

ii)



c) $F_{NET_c} = \frac{mv^2}{R} = N \Rightarrow \boxed{N = \frac{mv^2}{R}}$

d) $F_f = \mu N \Rightarrow F_f = \frac{\mu m v^2}{R} = m a_{tan} \Rightarrow |a_{tan}| = \boxed{\frac{\mu v^2}{R}}$

e)

$a_{tan} = -\frac{\mu v^2}{R} \Rightarrow \frac{dv}{dt} = -\frac{\mu v^2}{R} \Rightarrow \frac{dv}{v^2} = -\frac{\mu}{R} dt \Rightarrow$

$\int_{v_0}^{v} v^{-2} dv = -\frac{\mu}{R} \int_{t=0}^{t} dt \Rightarrow -v^{-1}\Big|_{v_0}^{v} = -\frac{\mu}{R} t \Rightarrow -\left(\frac{1}{v} - \frac{1}{v_0}\right) = -\frac{\mu}{R} t$

$\Rightarrow \frac{1}{v_0} - \frac{1}{v} = -\frac{\mu}{R} t \Rightarrow \frac{v_0}{v_0} - \frac{v_0}{v} = -\frac{v_0 \mu t}{R} \Rightarrow -\frac{v_0}{v} = -\frac{v_0 \mu t}{R} - 1 \Rightarrow$

$\frac{v_0}{v} = \frac{R}{R} + \frac{v_0 \mu t}{R} \Rightarrow \frac{v_0}{v} = \frac{R + v_0 \mu t}{R} \Rightarrow \frac{v}{v_0} = \frac{R}{R + v_0 \mu t} \Rightarrow$

$$\boxed{v = \frac{v_0 R}{R + v_0 \mu t}}$$

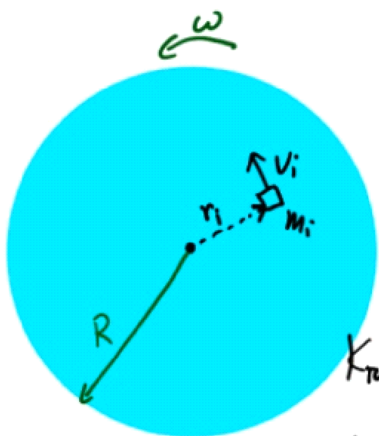
7.2 - Moment of Inertia

Wednesday, March 15, 2017 8:21 PM

Types of Inertia

- Inertial mass / translational inertia (M) is an object's ability to resist a linear acceleration
- Moment of Inertia / rotational inertia (I) is an object's resistance to a rational acceleration
- Objects that have most of their mass near their axis of rotation have smaller rotational inertias than objects with more mass farther from their axis of rotation

Kinetic Energy of a Rotating Disc



$$K_i = \frac{1}{2} m_i v_i^2 \xrightarrow{v=\omega r}$$

$$K_i = \frac{1}{2} m_i \omega^2 r_i^2$$

$$K_{TOT} = \sum_i K_i = \sum_i \frac{1}{2} m_i \omega^2 r_i^2 \Rightarrow$$

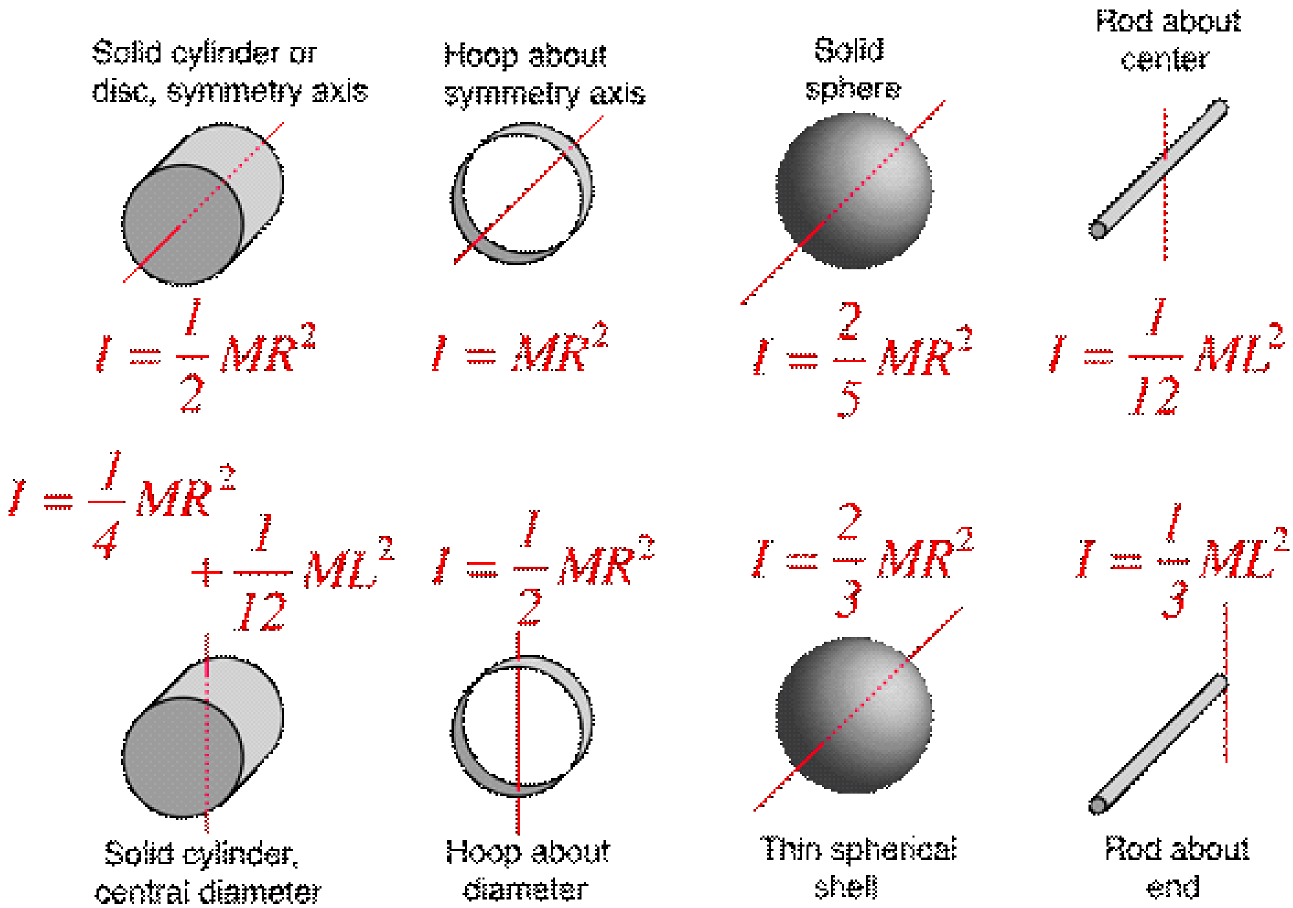
$$K_{TOT} = \frac{\omega^2}{2} \sum_i m_i r_i^2 \xrightarrow{I = \sum m r^2}$$

$$K_{TOT} = \frac{1}{2} I \omega^2$$

$$K = \frac{1}{2} m v^2$$

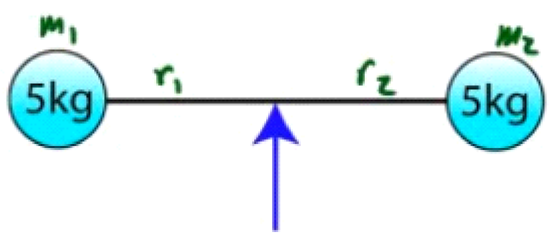
Calculating Moment of Inertia (I)

- $I = \sum mr^2 = \int r^2 dm$



Example 1: Point Masses

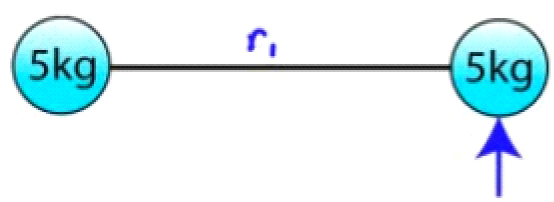
- Find the moment of inertia (I) of two 5-kg bowling balls joined by a meter-long rod of negligible mass when rotated about the center of the rod.
- Compare this to the moment of inertia of the object when rotated about one of the mass



$$I = \sum mr^2 = m_1r_1^2 + m_2r_2^2$$

$$I = 5(.5)^2 + 5(.5)^2$$

$$I = 2.5\ kg \cdot m^2$$

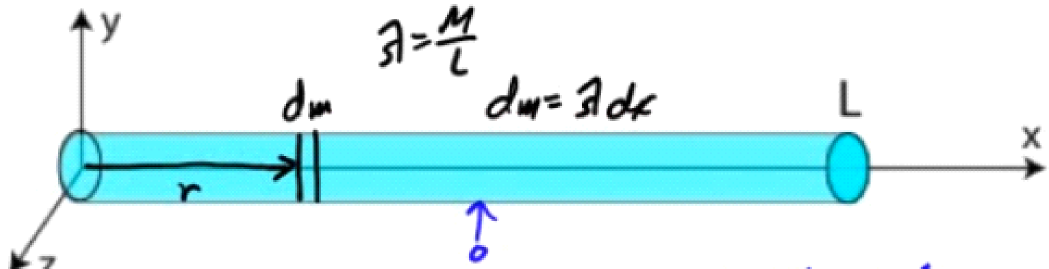


$$I = \sum mr^2 = m_1r_1^2 + m_2r_2^2$$

$$I = 5(1)^2 + 5(0)^2 = 5\ kg \cdot m^2$$

Example 2: Uniform Rod

- Find the moment of inertia of a uniform rod about its end and about its center



$\lambda = \frac{M}{L}$

$dm = \lambda dx$

about end

$$I = \int r^2 dm = \int_0^L x^2 \lambda dx \Rightarrow$$

$$I = \lambda \int_0^L x^2 dx = \lambda \frac{x^3}{3}\Big|_0^L = \lambda \frac{L^3}{3} \Rightarrow$$

$$I = \frac{M}{\cancel{L}} \frac{L^{\cancel{3}2}}{3} = \boxed{\frac{ML^2}{3}}$$

about center

$$I = \int r^2 dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \lambda dx \Rightarrow$$

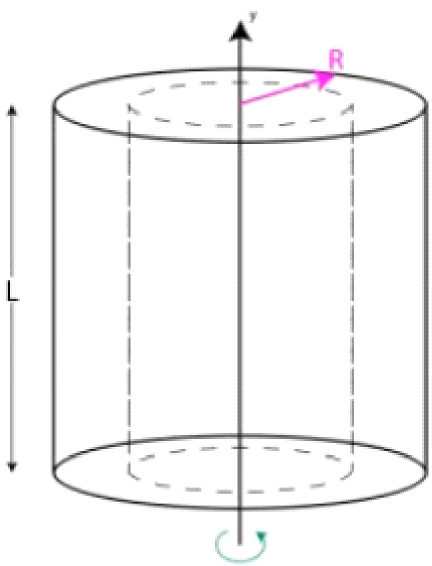
$$I = \lambda \frac{x^3}{3}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{\lambda}{3}\left(\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right)$$

$$I = \frac{\lambda}{3}\frac{L^3}{4} = \frac{M L^{\cancel{3}2}}{\cancel{L}\, 12} = \frac{1}{12} ML^2$$

Example 3: Solid Cylinder

- Find the moment of inertia of a uniform solid cylinder about its axis

$\rho = \frac{M}{V} = \frac{M}{\pi R^2 L}$ $\quad dm = 2\pi r L \rho dr$



$$I = \int r^2 dm = \int_{r=0}^{R} r^2 (2\pi r L \rho dr) \Rightarrow$$

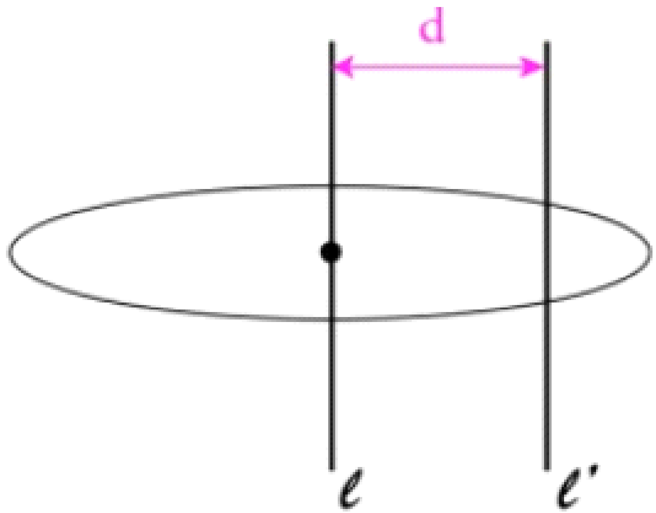
$$I = 2\pi\rho L \int_0^R r^3 dr = 2\pi\rho L \frac{r^4}{4}\Big|_0^R \Rightarrow$$

$$I = 2\pi\rho L \frac{R^4}{4} \quad \rho = \frac{M}{\pi R^2 L} \quad \frac{2\cancel{\pi} M \cancel{L} R^{\cancel{4}2}}{\cancel{\pi} \cancel{R^2} \cancel{L}\, \cancel{4}\, 2} \Rightarrow$$

$$\boxed{I = \frac{MR^2}{2}}$$

Parallel Axis Theorem (PAT)

- If the moment of inertia (I) of any object through an axis intersecting the center of mass of the object is I, you can find the moment of inertia around any axis parallel to the current axis of rotation (l')



$I_{l'} = I_l + md^2$

$$I = \int [(x' + x_{cm})^2 + (y' + y_{cm})^2]\,dm$$

$$= \int [(x'^2 + 2x'x_{cm} + x_{cm}{}^2) + (y'^2 + 2y'y_{cm} + y_{cm}{}^2)]\,dm$$

$$= \boxed{\int (x'^2 + y'^2)\,dm} + \boxed{\int (x_{cm}{}^2 + y_{cm}{}^2)\,dm}$$

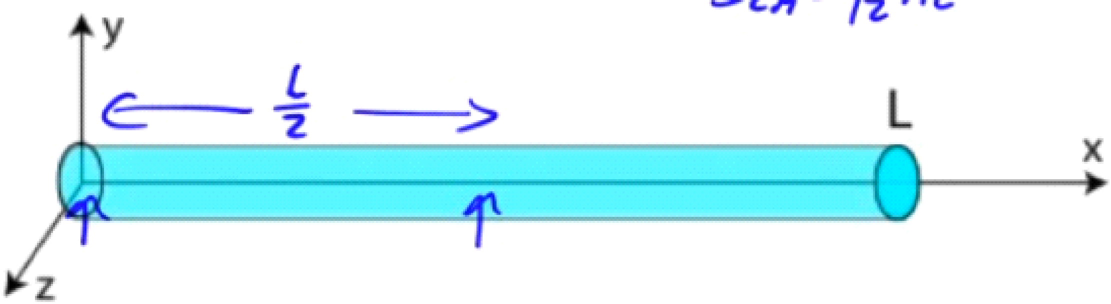
$$+ 2x_{cm} \int x'\,dm + 2y_{cm} \int y'\,dm$$

$$= \boxed{I_{CM}} + \boxed{MR^2}$$

Example 4: Calculating I Using PAT

- Find the moment of inertia of a uniform rod about its end

$$I_{cm} = \frac{1}{12}ML^2$$



$$I_{end} = I_{cm} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{12} + \frac{ML^2}{4} \Rightarrow$$

$$I_{end} = \frac{ML^2}{12} + \frac{3ML^2}{12} = \frac{4ML^2}{12} = \boxed{\frac{ML^2}{3}}$$

Example 5: Hollow Sphere

- Calculate the moment of inertia of a hollow sphere with a mass of 10 kg and a radius of 0.2 meter

$$I = \frac{2}{3}MR^2 \rightarrow \frac{2}{3}(10)(.2)^2 = \boxed{0.27\ kg \cdot m^2}$$

Example 6: Adjusting Moment of Inertia

- A uniform rod of length L has moment of inertia I_0 when rotated about its midpoint.
- A sphere of mass M is added to each end of the rod.
- What is the new moment of inertia of the rod/ball system?

$$I = I_0$$

$$I = I_0 + \Sigma m r^2 \Rightarrow$$

$$I = I_0 + M\left(\frac{L}{2}\right)^2 + M\left(\frac{L}{2}\right)^2$$

$$I = I_0 + \frac{ML^2}{4} + \frac{ML^2}{4}$$

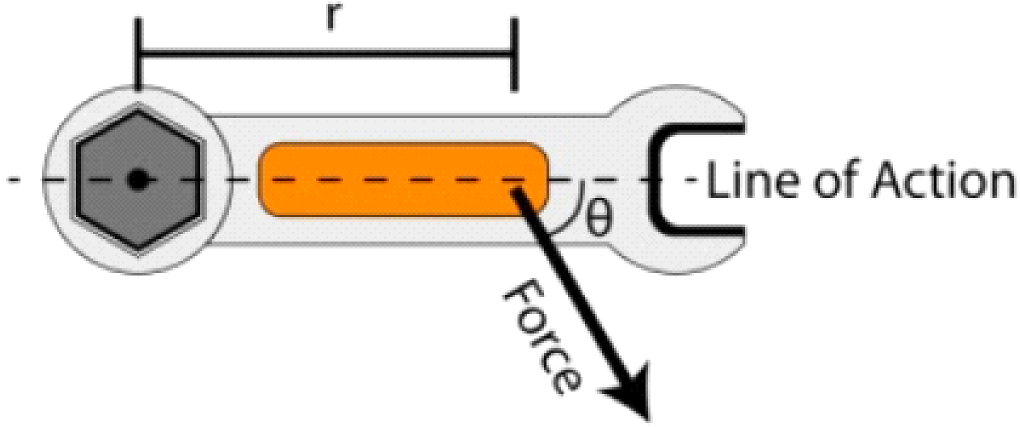
$$I = I_0 + \frac{ML^2}{2}$$

7.3 - Torque

Thursday, March 16, 2017 11:31 PM

Torque (τ)

- Torque is a force that causes an object to turn
- Torque must be perpendicular to the displacement to cause a rotation
- The further away the force is applied from the point of rotation, the more leverage you obtain, so this distance is known as the lever arm (r)



- $\vec{\tau} = \vec{r} \times \vec{F}$
- $|\vec{\tau}| = rF\sin\theta$

Direction of the Torque Vector

- The direction of the torque vector is perpendicular to both the position vector and the force vector
- You can find the direction using the right-hand rule. Point the fingers of your right hand in the direction of the line of action, and bend you fingers in the direction of the force
- You thumb then points in the direction of your torque
- Note that positive torques cause counter-clockwise rotation, and negative torques cause clockwise rotation

Newton's Second Law: Translational vs. Rotational

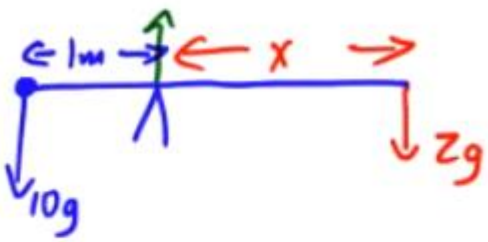
- $\vec{F}_{net} = m\vec{a}$
- $\vec{\tau} = I\vec{\alpha}$

Equilibrium

- Static Equilibrium implies that the net force and the net torque are zero, and the system is at rest
- Dynamic Equilibrium implies that the net force and the net torque are zero, and the system is moving at constant translational and rotational velocity

Example 1: See-Saw Problem

- A 10-kg tortoise sits on a see-saw 1 meter from the fulcrum.
- Where must a 2-kg hare sit in order to maintain static equilibrium?
- What is the force on the fulcrum?



$$\tau_{NET} = I\alpha = 0 \Rightarrow$$

$$(10g)(1) - (2g)x = 0 \Rightarrow$$

$$10g - 2gx = 0$$

$$10g = 2gx$$

$$\boxed{x = 5m}$$

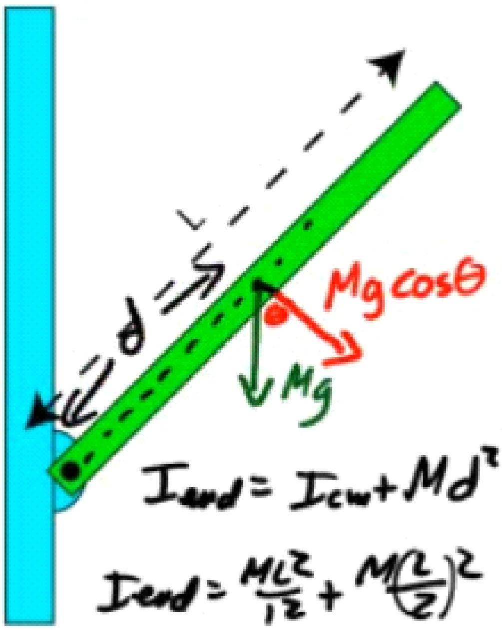
$$F_{NET} = ma = 0$$

$$-10g - 2g + F_{fulcrum} = 0$$

$$F_{fulcrum} = 12g = 12(10\tfrac{m}{s^2}) = \boxed{120N}$$

Example 2: Beam Problem

- A beam of mass M and Length L has a moment of inertia about its center of $\frac{1}{12}ML^2$. The beam is attached to a frictionless hinge at an angle of 45° and allowed to swing freely.
- Find the beam's angular acceleration



$$I_{end} = I_{cm} + Md^2$$

$$I_{end} = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$$

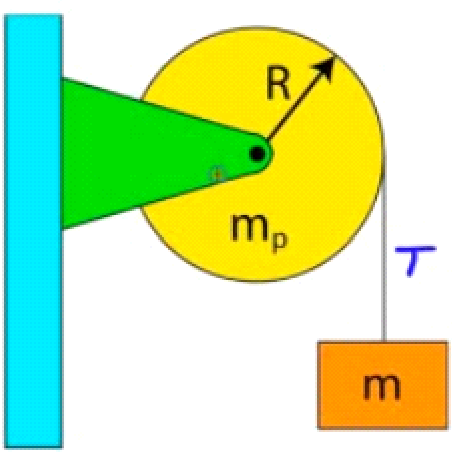
$$= \frac{1}{3}ML^2$$

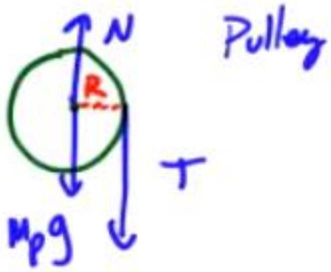
$$\tau_{NET} = I\alpha \Rightarrow -Mg\cos\theta\left(\frac{L}{2}\right) = I\alpha \Rightarrow$$

$$\alpha = \frac{-Mg\cos\theta L}{2I} = \frac{-\cancel{M}g\cos\theta \cancel{L}\,3}{2\,1\,\cancel{M}L^{\cancel{2}}} = \boxed{\frac{-3g\cos\theta}{2L}}$$

Example 3: Pulley with Mass

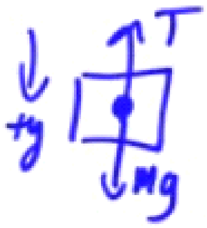
- A light string attached to a mass m is wrapped around a pulley of mass m_p and radius R. Find the acceleration of the mass





$$\tau_{net} = I\alpha \xrightarrow[I=\frac{1}{2}m_pR^2]{\tau=RT} RT = \frac{1}{2}m_pR^2\alpha \Rightarrow$$

$$T = \frac{1}{2}m_pR\alpha \xrightarrow[a=R\alpha]{\alpha=\frac{a}{R}} T = \frac{1}{2}m_pa$$



$$mg - T = ma$$

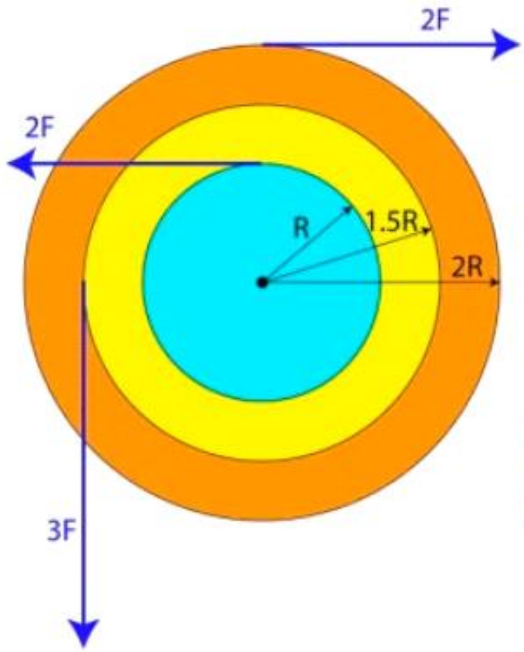
$$mg - \frac{1}{2}m_pa = ma$$

$$mg = \frac{1}{2}m_pa + ma \Rightarrow$$

$$\boxed{a = \left(\frac{mg}{m + \frac{m_p}{2}}\right)}$$

Example 4: Net Torque

- A system of three wheels fixed to each other is free to rotate about an axis through its center. Forces are exerted on the wheels as shown. What is the magnitude of the net torque on the wheels?



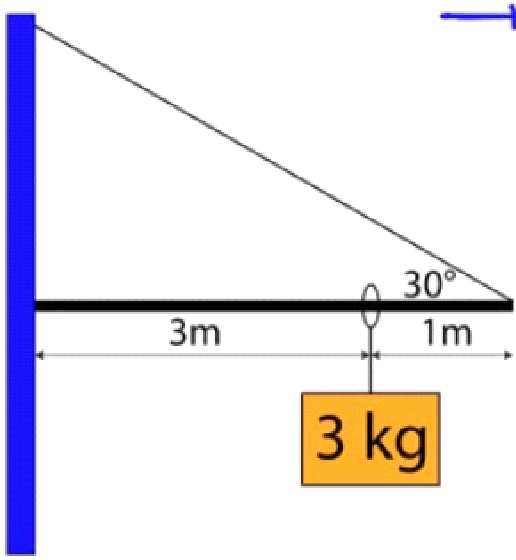
$$\tau_{NET} = -(2F)(2R) + 2F(1R) + 3F(1.5R) \Rightarrow$$

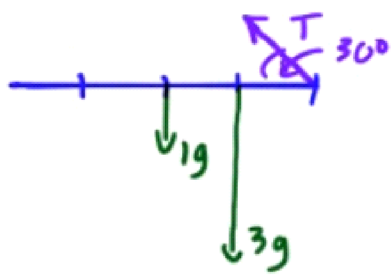
$$\tau_{NET} = -4FR + 2FR + 4.5FR$$

$$\boxed{\tau_{NET} = 2.5FR}$$

Example 5: Café Sign

- A 3-kg café sign is hung from a 1-kg horizontal pole as shown. A wire is attached to prevent the sign from rotating.
- Find the tension in the wire

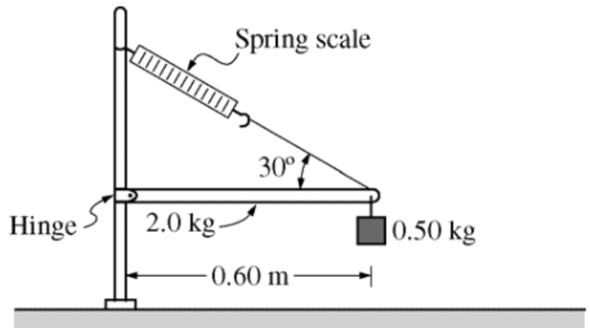




$$\tau_{NET} = 0 \Rightarrow T\sin 30°(4) - 3g(3) - 1g(2) \Rightarrow$$

$$T = \frac{11g}{4\sin 30°} = 54N$$

2008 Free Response Question 2



Mech. 2.

The horizontal uniform rod shown above has length 0.60 m and mass 2.0 kg. The left end of the rod is attached to a vertical support by a frictionless hinge that allows the rod to swing up or down. The right end of the rod is supported by a cord that makes an angle of 30° with the rod. A spring scale of negligible mass measures the tension in the cord. A 0.50 kg block is also attached to the right end of the rod.

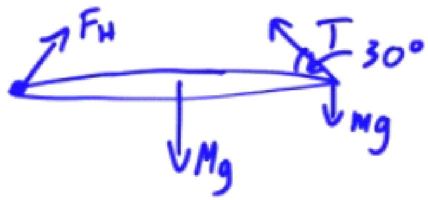
(a) On the diagram below, draw and label vectors to represent all the forces acting on the rod. Show each force vector originating at its point of application.

(b) Calculate the reading on the spring scale.

(c) The rotational inertia of a rod about its center is $\frac{1}{12}ML^2$, where M is the mass of the rod and L is its length. Calculate the rotational inertia of the rod-block system about the hinge.

(d) If the cord that supports the rod is cut near the end of the rod, calculate the initial angular acceleration of the rod-block system about the hinge.

a)



$M = 2\text{ kg}$
$m = .5\text{ kg}$
$L = .6\text{ m}$

b) $\tau_{net} = 0 \Rightarrow TL\sin 30° - mgL - Mg\frac{L}{2} = 0 \Rightarrow$

$$T = \frac{gL(m+\frac{M}{2})}{L\sin 30°} \Rightarrow T = 2g\left(m+\frac{M}{2}\right) = 2(10)(.5+1) = \boxed{30\text{ N}}$$

c) $I_S = I_{rod} + I_{block}$ $\qquad$ $I_{rod_{cM}} = \frac{1}{12} ML^2$ $\qquad$ Use P.A.T.

$$I_{r_H} = I_{r_{cM}} + Md^2$$

$$= \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{12} + \frac{ML^2}{4}$$

$$I_{r_H} = \frac{ML^2}{3} \qquad I_{block} = mL^2$$

$$I_{system} = \frac{1}{3} ML^2 + mL^2 = L^2\left(\frac{M}{3} + m\right) = (0.6)^2\left(\frac{3}{3} + .5\right) =$$

$$\boxed{0.42\ kg \cdot m^2}$$

d) $$\tau_{NET} = I\alpha \Rightarrow \alpha = \frac{\tau_{NET}}{I} = \frac{mgL + Mg\frac{L}{2}}{I} =$$

$$\frac{gL}{I}\left(m + \frac{M}{2}\right) = \frac{(10)(.6)}{.42}(.5 + 1) \Rightarrow$$

$$\boxed{\alpha = 21.4\ rad/s^2}$$

7.4 - Rotational Dynamics

Thursday, March 16, 2017 11:32 PM

Conservational of Energy

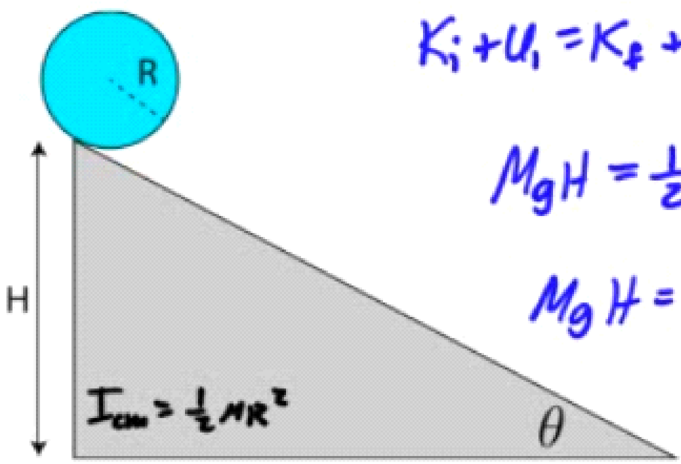
$$K_{translational} = \frac{1}{2}mv^2$$

$$K_{rotational} = \frac{1}{2}I\omega^2$$

$$K_{total} = K_{translational} + K_{rotational}$$

Example 1: Disc Rolling Down an Incline

- Find the speed of a disc of radius R which starts at rest and rolls down an incline of height H



$$K_i + U_i = K_f + U_f \xrightarrow[U_f=0]{K_i=0} U_i = K_f \Rightarrow$$

$$MgH = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2 \xrightarrow{I=\frac{1}{2}MR^2}$$

$$MgH = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2 \Longrightarrow$$

$$gH = \frac{v^2}{2} + \frac{1}{4}R^2\omega^2 \xrightarrow[R^2\omega^2=v^2]{v=\omega R}$$

$$gH = \frac{v^2}{2} + \frac{v^2}{4} \Rightarrow gH = \frac{3}{4}v^2 \Rightarrow v^2 = \frac{4gH}{3} \Rightarrow$$

$$\boxed{v = \sqrt{\frac{4}{3}gH}}$$

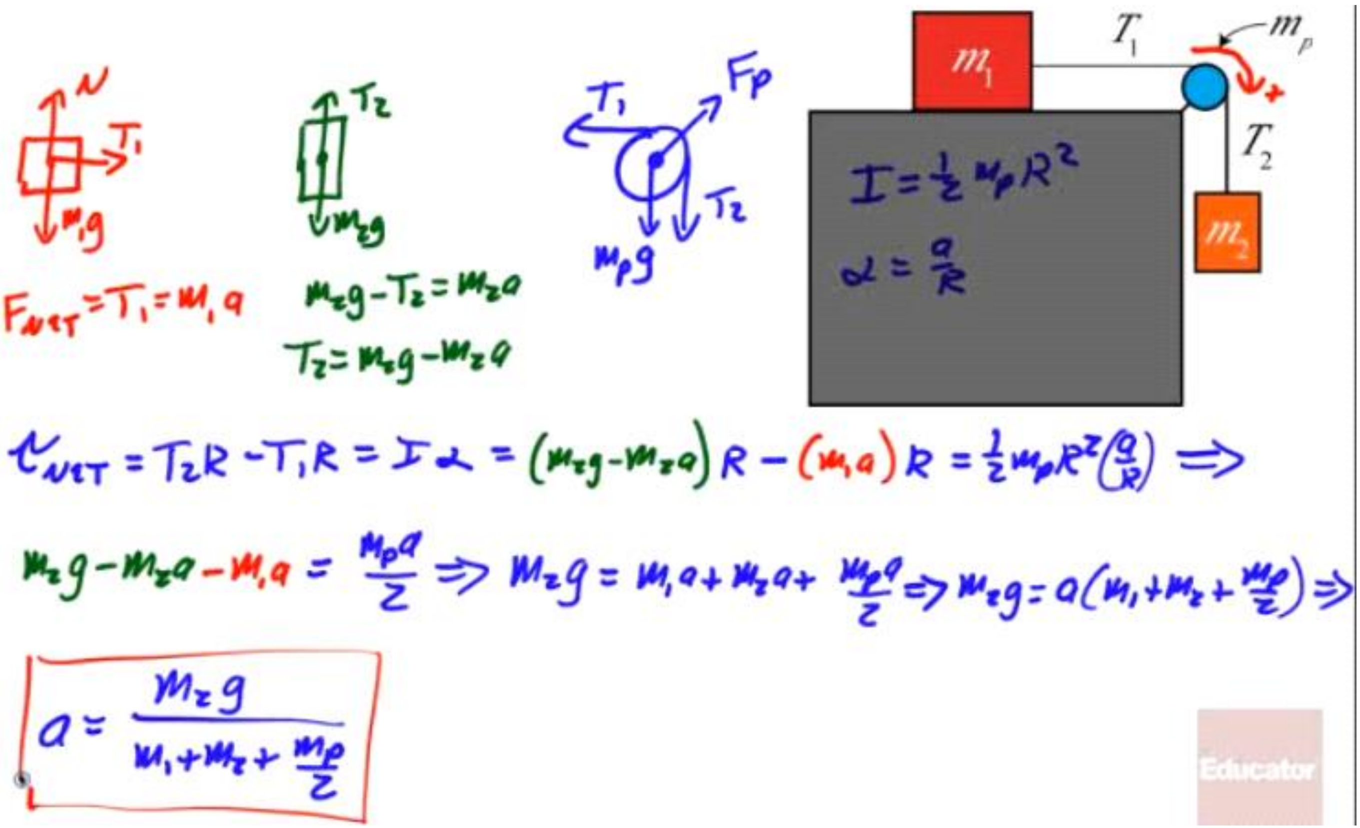
Rotational Dynamics

$$\vec{F}_{net} = m\vec{a}$$

$$\vec{\tau}_{net} = I\vec{\alpha}$$

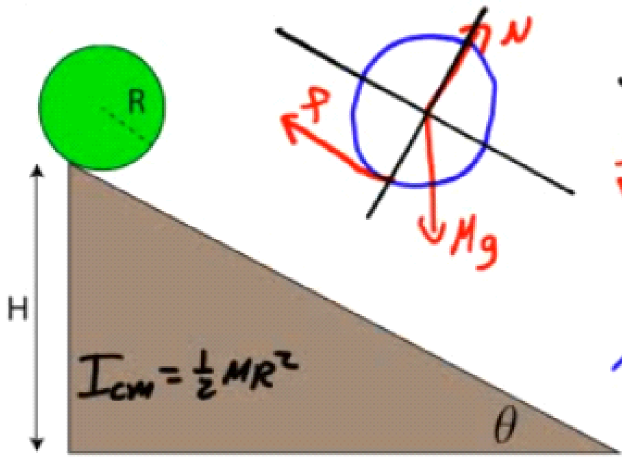
Example 2: Strings with Massive Pulleys

- Two blocks are connected by a light string over a pulley of mass m_p and radius R.
- Find the acceleration of mass m_2 if m_1 stis on a frictionless surface

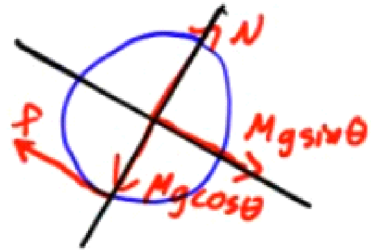


Example 3: Rolling Without Slipping

- A disc of radius R rolls down an incline of angle θ without slipping.
- Find the force of friction on the disc



$I_{cm} = \frac{1}{2}MR^2$



$N = Mg\cos\theta$

$Mg\sin\theta - f = Ma$

$\tau_{Net} = fR = I\alpha \xrightarrow{I=\frac{1}{2}MR^2} fR = \frac{MR^2\alpha}{2} \Rightarrow$

$fR = \frac{MRa}{2} \Rightarrow f = \frac{Ma}{2} = \frac{Mg\sin\theta - f}{2} \Rightarrow$

$2f = Mg\sin\theta - f \Rightarrow 3f = Mg\sin\theta \Rightarrow$

$$\boxed{f = \frac{Mg\sin\theta}{3}}$$

Example 4: Rolling with Slipping

- A bowling ball of mass M and radius R skids horizontally down the alley with an initial velocity of v_0. Find the distance the ball skids before rilling given a coefficient of kenetic friction μ_k

$F_{NET_x} = -f_k = Ma \qquad N = Mg$

$-f_k = -\mu_k N = -\mu_k Mg = Ma \Rightarrow a = -\mu_k g$

$v = v_0 + at \xrightarrow{a=-\mu_k g} \boxed{v = v_0 - \mu_k g t}$

$\tau_{Net} = f_k R = I\alpha \xrightarrow[I=\frac{2}{5}MR^2]{f_k=\mu_k Mg} \mu_k MgR = \frac{2}{5}MR^2\alpha \Rightarrow \mu_k g = \frac{2}{5}R\alpha \Rightarrow$

$\alpha = \frac{5\mu_k g}{2R}$

$\omega = \omega_0 + \alpha t \xrightarrow[\omega_0=0]{\alpha=\frac{5\mu_k g}{2R}} \boxed{\omega = \frac{5\mu_k g}{2R}t}$

Stops slipping when $v = R\omega$

$$v_0 - \mu_k g t = R\left(\frac{5\mu_k g}{2R}\right)t \Rightarrow v_0 - \mu_k g t = \frac{5\mu_k g}{2}t \Rightarrow$$

$$v_0 = \mu_k g t + \frac{5}{2}\mu_k g t \Rightarrow v_0 = \frac{7}{2}\mu_k g t \Rightarrow t = \frac{2v_0}{7\mu_k g}$$

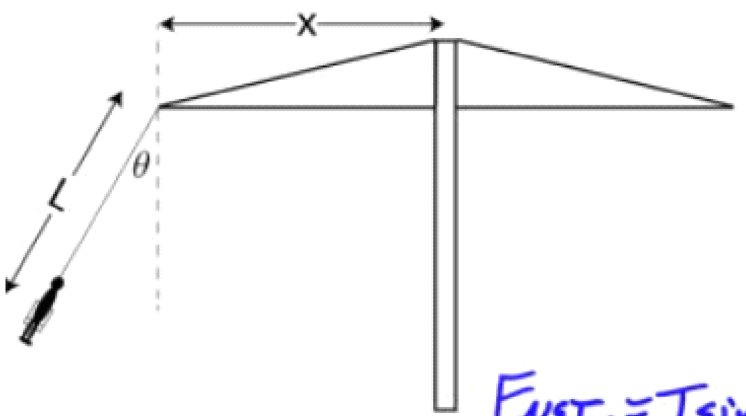
$$\Delta x = v_0 t + \frac{1}{2}at^2 = v_0\left(\frac{2v_0}{7\mu_k g}\right) + \frac{1}{2}(-\mu_k g)\left(\frac{2v_0}{7\mu_k g}\right)^2 \Rightarrow$$

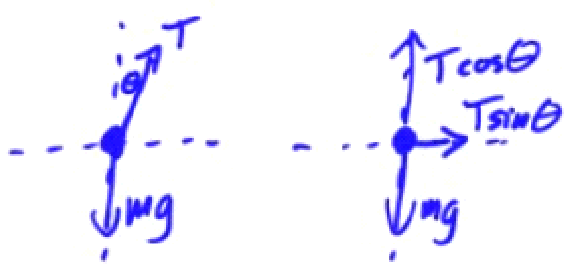
$$\Delta x = \frac{2v_0^2}{7\mu_k g} - \frac{1}{2}\mu_k g\frac{4v_0^2}{49\mu_k^2 g^2} = \frac{2v_0^2}{7\mu_k g} - \frac{2v_0^2}{49\mu_k g} = \frac{14v_0^2}{49\mu_k g} - \frac{2v_0^2}{49\mu_k g} \Rightarrow$$

$$\boxed{\Delta x = \frac{12v_0^2}{49\mu_k g}}$$

Example 5: Amusement Park Swing

- An amusement park ride of radius x allows children to sit in a spinning swing held by a cable of length L.
- At maximum angular speed, the cable makes an angle of θ with the vertical as shown in the diagram below
- Determine the maximum angular speed of the rider in terms of g, θ, x and L.



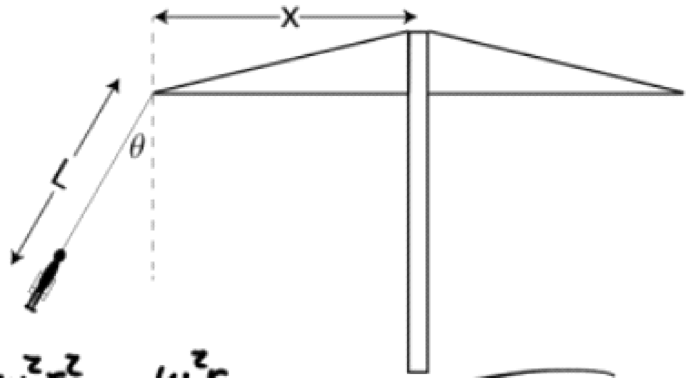


$$F_{NET_x} = T\sin\theta = ma = \frac{mv^2}{r}$$

$$F_{NET_y} = T\cos\theta - mg = 0 \Rightarrow T\cos\theta = mg$$

$$\frac{T\sin\theta}{T\cos\theta} = \frac{mv^2/r}{mg}$$

$$\tan\theta = \frac{v^2}{gr}$$

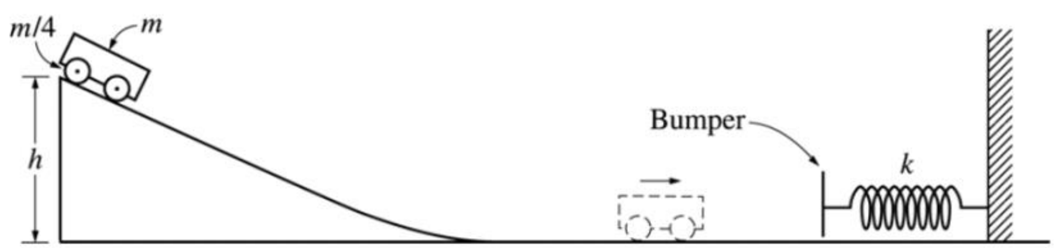


- $\tan\theta = \frac{v^2}{gr} \xRightarrow{v=\omega r} \tan\theta = \frac{\omega^2 r^2}{gr} = \frac{\omega^2 r}{g} \Rightarrow \omega = \sqrt{\frac{g\tan\theta}{r}}$

- $r = x + L\sin\theta$

$$\omega = \sqrt{\frac{g\tan\theta}{r}} = \boxed{\sqrt{\frac{g\tan\theta}{x+L\sin\theta}}}$$

2002 Free Response Question 2



Mech 2.

The cart shown above is made of a block of mass m and four solid rubber tires each of mass $m/4$ and radius r. Each tire may be considered to be a disk. (A disk has rotational inertia $\frac{1}{2}ML^2$, where M is the mass and L is the radius of the disk.) The cart is released from rest and rolls without slipping from the top of an inclined plane of height h. Express all algebraic answers in terms of the given quantities and fundamental constants.

(a) Determine the total rotational inertia of all four tires.

(b) Determine the speed of the cart when it reaches the bottom of the incline.

(c) After rolling down the incline and across the horizontal surface, the cart collides with a bumper of negligible mass attached to an ideal spring, which has a spring constant k. Determine the distance x_m the spring is compressed before the cart and bumper come to rest.

(d) Now assume that the bumper has a non-neglible mass. After the collision with the bumper, the spring is compressed to a maximum distance of about 90% of the value of x_m in part (c). Give a reasonable explanation for this decrease.

a) $I_{tires} = 4 I_{tire} = 4\left(\frac{1}{2}MR^2\right) = 4\left(\frac{1}{2}\frac{m}{4}r^2\right) = \boxed{\frac{1}{2}mr^2}$

b) $U_i = K_r + K_T \Rightarrow 2mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}(2m)v^2 \Rightarrow$

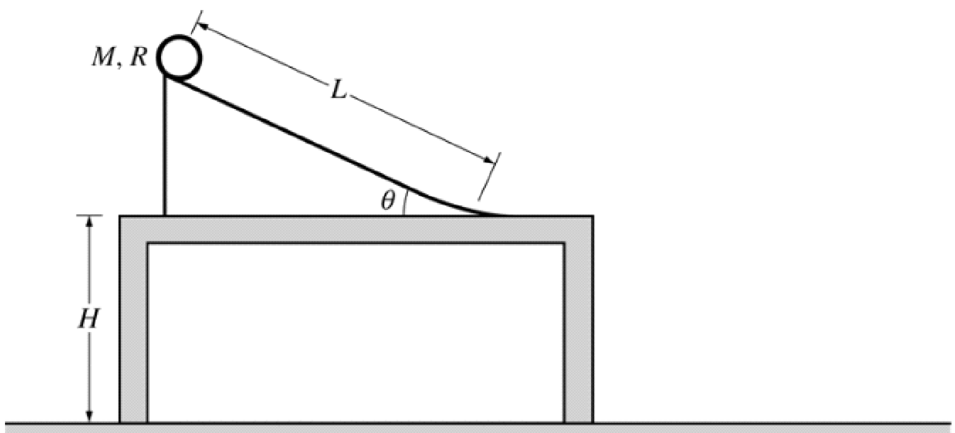
$2mgh = \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2 + \frac{1}{2}(2m)v^2 \xRightarrow[\omega = v/r]{v=\omega r} 2mgh = \frac{1}{4}mr^2\frac{v^2}{r^2} + mv^2 \Rightarrow$

$2mgh = \frac{1}{4}mv^2 + mv^2 \Rightarrow 2mgh = \frac{5}{4}mv^2 \Rightarrow 2gh\left(\frac{4}{5}\right) = v^2 \Rightarrow v^2 = \frac{8gh}{5} \Rightarrow$

$\boxed{v = \sqrt{\frac{8gh}{5}}}$

c) $U_i = U_s \Rightarrow 2mgh = \frac{1}{2}kx_m^2 \Rightarrow \frac{4mgh}{k} = x_m^2 \Rightarrow \boxed{x_m = 2\sqrt{\frac{mgh}{k}}}$

2006 Free Response Question 3



Mech 3.

A thin hoop of mass M, radius R, and rotational inertia MR^2 is released from rest from the top of the ramp of length L above. The ramp makes an angle θ with respect to a horizontal tabletop to which the ramp is fixed. The table is a height H above the floor. Assume that the hoop rolls without slipping down the ramp and across the table. Express all algebraic answers in terms of given quantities and fundamental constants.

(a) Derive an expression for the acceleration of the center of mass of the hoop as it rolls down the ramp.

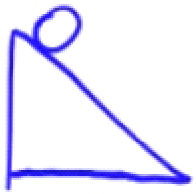
(b) Derive an expression for the speed of the center of mass of the hoop when it reaches the bottom of the ramp.

(c) Derive an expression for the horizontal distance from the edge of the table to where the hoop lands on the floor.

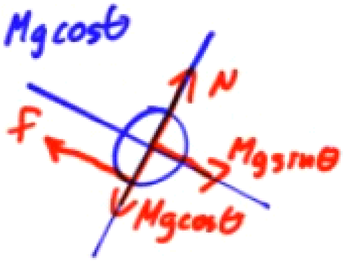
(d) Suppose that the hoop is now replaced by a disk having the same mass M and radius R. How will the distance from the edge of the table to where the disk lands on the floor compare with the distance determined in part (c) for the hoop?

____ Less than ____ The same as ____ Greater than

Briefly justify your response.



$N = Mg\cos\theta$



$\tau_{NET} = I\alpha$

$-fR = -I\alpha$

$fR = I\alpha$

$F_{NET_x} = Mg\sin\theta - f = Ma_{cm}$

$a_{cm} = \alpha R \Rightarrow \alpha = \frac{a_{cm}}{R}$ $\Big\} \Rightarrow$

$I = MR^2$

$Mg\sin\theta - Ma = Ma$

$g\sin\theta = 2a$

$\boxed{a = \frac{g\sin\theta}{2}}$

$fR = \frac{MR^2 a}{R} \Rightarrow fR = MRa$

$f = Ma$

b) $v_f^2 = v_0^2 + 2a\Delta x \Rightarrow v_f^2 = 2a\Delta x \Rightarrow v_f = \sqrt{2a\Delta x} \Rightarrow$

$\boxed{v_f = \sqrt{gl\sin\theta}}$

c) $\underline{\text{Vert}}$

$v_i = 0$

$\Delta y = H$

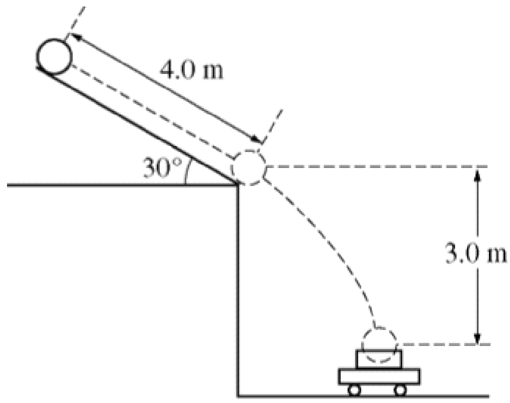
$a_y = g$

$\Delta y = \cancel{v_0 t}^{\,0} + \frac{1}{2}a_y t^2 \Rightarrow \Delta y = \frac{1}{2}a_y t^2 \Rightarrow t = \sqrt{\frac{2\Delta y}{a}} = \sqrt{\frac{2H}{g}}$

Horz: $\Delta x = v_x t = \sqrt{gl\sin\theta}\left(\sqrt{\frac{2H}{g}}\right) = \boxed{\sqrt{2Hl\sin\theta}}$

d) $I_{disk} = \frac{1}{2}MR^2$ Greater than

2010 Free Response Question 2

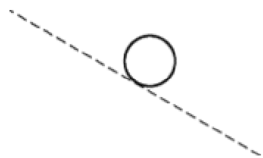


Note: Figure not drawn to scale.

Mech. 2.

A bowling ball of mass 6.0 kg is released from rest from the top of a slanted roof that is 4.0 m long and angled at $30°$, as shown above. The ball rolls along the roof without slipping. The rotational inertia of a sphere of mass M and radius R about its center of mass is $\frac{2}{5}MR^2$.

(a) On the figure below, draw and label the forces (not components) acting on the ball at their points of application as it rolls along the roof.

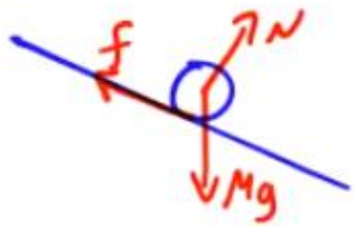


(b) Calculate the force due to friction acting on the ball as it rolls along the roof. If you need to draw anything other than what you have shown in part (a) to assist in your solution, use the space below. Do NOT add anything to the figure in part (a).

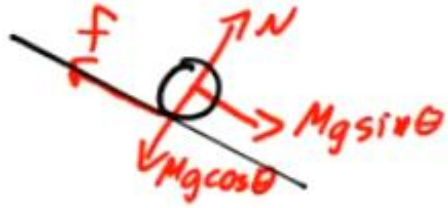
(c) Calculate the linear speed of the center of mass of the ball when it reaches the bottom edge of the roof.

(d) A wagon containing a box is at rest on the ground below the roof so that the ball falls a vertical distance of 3.0 m and lands and sticks in the center of the box. The total mass of the wagon and the box is 12 kg. Calculate the horizontal speed of the wagon immediately after the ball lands in it.

a)



b)



$$① \; F_{NET_x} = Mg\sin\theta - f = Ma$$

$$② \; \tau_{NET} = I\alpha \Rightarrow fR = I\alpha \Rightarrow fR = \tfrac{2}{5}MR^2\alpha$$

$$③ \; a = R\alpha$$

$$②+③ \; fR = \tfrac{2}{5}MR^2\tfrac{a}{R} \Rightarrow fR = \tfrac{2}{5}MRa \Rightarrow Ma = \frac{5f}{2} \; ④$$

$$①+④ \; Mg\sin\theta - f = \frac{5f}{2} \Rightarrow Mg\sin\theta = \frac{7f}{2} \Rightarrow f = \frac{2Mg\sin\theta}{7} \Rightarrow$$

$$f = \frac{2(6)(10)\sin 30^\circ}{7} = \boxed{8.5\,N}$$

$$c) \; U_g = K_t + K_r \Rightarrow Mgh = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2 \Rightarrow Mgd\sin\theta = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}\left(\tfrac{2}{5}MR^2\right)\omega^2$$

$v=\omega R \Rightarrow v^2=\omega^2R^2$

$$Mgd\sin\theta = \tfrac{1}{2}Mv^2 + \tfrac{1}{5}Mv^2 = \tfrac{7}{10}Mv^2 \Rightarrow v^2 = \frac{10Mgd\sin\theta}{7M} \Rightarrow$$

$$v^2 = \frac{10gd\sin\theta}{7} \Rightarrow v\sqrt{\frac{10(9.8)(4)(\sin 30^\circ)}{7}} = \boxed{5.29\,\tfrac{m}{s}}$$

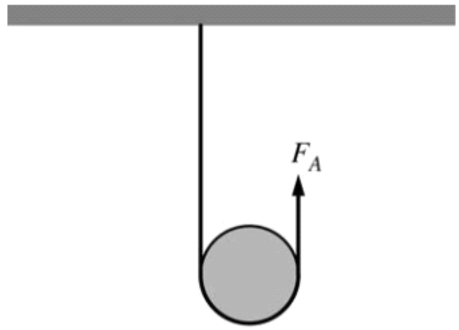
$$d) \; P_b = P_a \Rightarrow M_i v_i = M_f v_f \Rightarrow v_f = \frac{M_i}{M_f}v_i \Rightarrow \quad v_i = \left(5.29\tfrac{m}{s}\right)\cos 30^\circ = 4.58$$

$M_i = 6\,kg$

$M_f = 18\,kg$

$$v_f = \left(\frac{6\,kg}{18\,kg}\right)\left(4.58\tfrac{m}{s}\right) = \boxed{1.53\,\tfrac{m}{s}}$$

2013 Free Response Question 3



Note: Figure not drawn to scale.

Mech 3.

A disk of mass $M = 2.0$ kg and radius $R = 0.10$ m is supported by a rope of negligible mass, as shown above. The rope is attached to the ceiling at one end and passes under the disk. The other end of the rope is pulled upward with a force F_A. The rotational inertia of the disk around its center is $MR^2/2$.

(a) Calculate the magnitude of the force F_A necessary to hold the disk at rest.

At time $t = 0$, the force F_A is increased to 12 N, causing the disk to accelerate upward. The rope does not slip on the disk as the disk rotates.

(b) Calculate the linear acceleration of the disk.

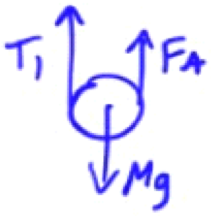
(c) Calculate the angular speed of the disk at $t = 3.0$ s.

(d) Calculate the increase in total mechanical energy of the disk from $t = 0$ to $t = 3.0$ s.

(e) The disk is replaced by a hoop of the same mass and radius. Indicate whether the linear acceleration of the hoop is greater than, less than, or the same as the linear acceleration of the disk.

__ Greater than __ Less than __ The same as

Justify your answer.

a) 

$T_1 + F_A = Mg \Rightarrow F_A = Mg - T_1$

$\tau_{net} = 0 \Rightarrow T_1 R - F_A R = 0 \Rightarrow T_1 = F_A$

$F_A = Mg - T_1 \Rightarrow F_A = Mg - F_A \Rightarrow F_A = \frac{Mg}{2} = \boxed{10\,N}$

b) $T_1 + F_A - Mg = Ma$ $\quad \tau_{net} = F_A R - T_1 R = I\alpha \xrightarrow[\alpha = \frac{a}{R}]{I = \frac{MR^2}{2}} F_A R - T_1 R = \frac{MR^2}{2}\frac{a}{R} \Rightarrow$

$F_A - T_1 = \frac{Ma}{2}$

$T_1 + F_A - Mg = Ma$

$F_A - T_1 = Ma/2$

$2F_A = \frac{3Ma}{2} + Mg \Rightarrow \frac{3Ma}{2} = 2F_A - Mg \Rightarrow a = \frac{2}{3M}(2F_A - Mg) \Rightarrow$

$a = \frac{4F_A}{3M} - \frac{2g}{3} = \frac{4(12)}{3(2)} - \frac{2(10)}{3} = \frac{48}{6} - \frac{20}{3} = \frac{8}{6} = \boxed{\frac{4}{3}\,m/s^2}$

c) $\omega = \omega_0 + \alpha t \xrightarrow{\alpha = \frac{a}{r}} \omega = \frac{at}{r} = \frac{(4/3)(3)}{.1} = \boxed{40\,\frac{rad}{s}}$

d) $\Delta h = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{4}{3}\right)(3)^2 = \frac{4}{6}\cdot 9 = 6\,m$ $\quad \Delta U_g = mg\Delta h = 2(10)(6) = \underline{120\,J}$

$\Delta K_r = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{MR^2}{2}\right)(40)^2 = \frac{1}{2}\left(\frac{2\cdot .1\cdot .1}{2}\right)(1600) = \underline{8\,J}$

$\Delta K_t = \frac{1}{2}mv^2 = \frac{1}{2}M(\omega R)^2 = \frac{1}{2}(2)(40)^2(.1)^2 = \underline{16\,J}$

$\Delta E_T = \boxed{144\,J}$

7.5 - Angular Momentum

Thursday, March 16, 2017 11:32 PM

Linear Momentum

- Momentum is a vector describing how difficult it is to stop a moving object
- Total momentum is the sum of individual momenta
- $\vec{p} = m\vec{v}$
- Units are kg·m/s or N·s

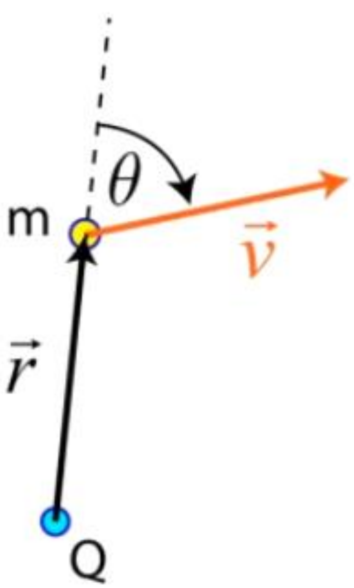
Angular Momentum

- Angular momentum ($\vec{L}$) is a vector describing how difficult it is to stop a rotating object
- Total angular momentum is the sum of individual angular momenta
- A mass with velocity $\vec{v}$ moving at some position $\vec{r}$ about point Q has angular momentum $\vec{L}_Q$
- Units are kg·m^2/s

Calculating Angular Momentum

- $\vec{L}_Q = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = (\vec{r} \times \vec{v})m$

direction by RHR ⊗



$|\vec{L}_Q| = mvr\sin\theta \xrightarrow{v=\omega r}$

$|\vec{L}_Q| = m\omega r^2$

$|\vec{L}_Q| = mr^2\omega$

Spin Angular Momentum

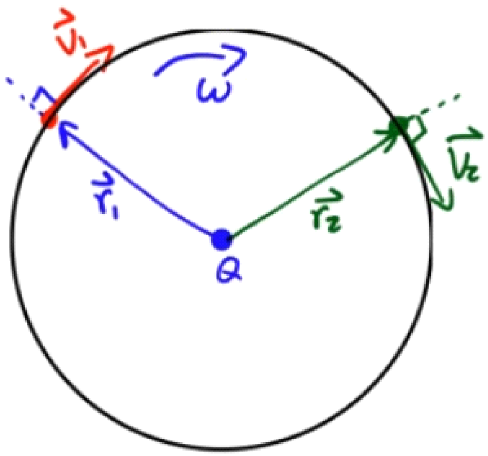
- For an object rotating about its center of mass

- $\vec{L} = I\vec{\omega}$

- This is known as an object's spin angular momentum
- Spin angular momentum is constant regardless of your reference point

Example 1: Object in Circular Orbit

- Find the angular momentum of a planet orbiting the sun. Assume a perfectly circular orbit

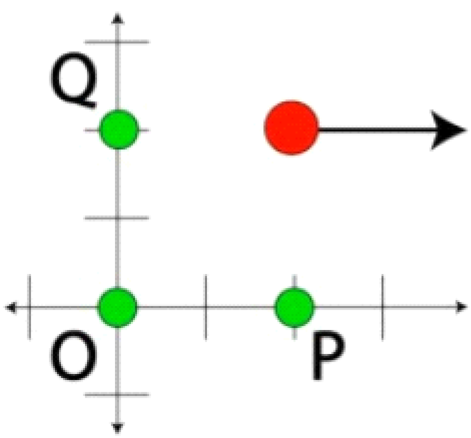


$$\vec{L}_Q = \vec{r}\times\vec{p} = \vec{r}\times m\vec{v} = (\vec{r}\times\vec{v})m$$

$$|\vec{L}_Q| = mvr\sin\theta = mvr$$

$\otimes$

Example 2: Angular Momentum of a Point Particle



- Find the angular momentum for a 5-kg point particle located at (2,2) with a velocity of 2 m/s east
- About point O at (0,0)

$$|\vec{L}_O| = mvr\sin\theta = (5)(2)(2\sqrt{2})\left(\frac{\sqrt{2}}{2}\right) = 20\ \frac{kg\cdot m^2}{s}$$

- About point P at (2,0)

$$|\vec{L}_P| = mvr\sin\theta = (5)(2)(2)\sin 90^\circ = 20\ \frac{kg\cdot m^2}{s}$$

- About point Q at (0,2)

$$|\vec{L}_Q| = mvr\sin\theta = (5)(2)(2)\sin\phi = 0$$

Angular Momentum and Net Torque

$$\vec{L}_Q = \vec{r}\times\vec{p} \xrightarrow{\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}} \frac{d\vec{L}_Q}{dt} = \overset{\vec{v}}{\frac{d\vec{r}}{dt}}\times\vec{p} + \vec{r}\times\overset{F}{\frac{d\vec{p}}{dt}} \Rightarrow$$

$$\frac{d\vec{L}_Q}{dt} = \vec{v}\times\vec{p} + \vec{r}\times\vec{F} \xrightarrow{\vec{v}\times\vec{p}=0} \frac{d\vec{L}_Q}{dt} = \vec{r}\times\vec{F} \xrightarrow{\vec{r}\times\vec{F}=\vec{\tau}}$$

$$\frac{d\vec{L}_Q}{dt} = \vec{\tau}_Q$$

Conservation of Angular Momentum

- Spin angular momentum, the product of an object's moment of inertia and its angular velocity about the center of mass, is conserved in a closed system with no external net toques applied
  - $\vec{L} = I\vec{\omega}$

Example 3: Ice Skater Problem

- An ice skater spins with a specific angular velocity. She brings her arms and legs closer to her body, reducing her moment of inertia to half its original value. What happens to her angular velocity? What happens to her rotational kinetic energy?

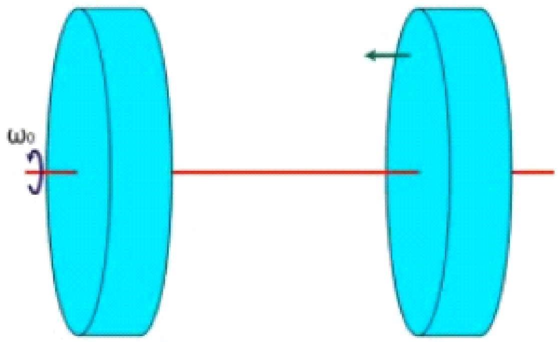
$\underline{L} = I\omega$ 　　ω doubles

$$K = \frac{1}{2}\underset{\text{half}}{I}\,\underset{\text{doble}}{\omega^2}$$

Doble K

Example 4: Combining Spinning Discs

- A disc with moment of inertia 1 km·m^2 spins about an axle through its center of mass with angular velocity 10 rad/s. An identical disc which is not rotating is slide along the axle until it makes contact with the first disc.
- If the two discs stick together, what is their combined angular velocity?



$$L_i = L_f$$

$$I_o \omega_o = I_f \omega_f$$

$$\omega_f = \frac{I_o \omega_o}{I_f} = \frac{(1\ \text{kg}\cdot\text{m}^2)(10\ \text{rad/s})}{2\ \text{kg}\cdot\text{m}^2}$$

$$\omega_f = 5\ \frac{\text{rad}}{\text{s}}$$

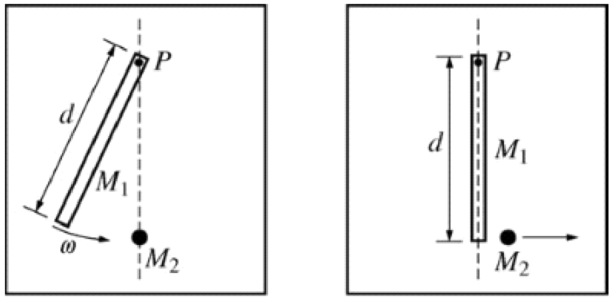
Example 5: Catching While Rotating

- Angelina spins on a rotating pedestal with an angular velocity of 8 radians per second.
- Bob throws her an exercise ball, which increases her moment of inertia from 2 kg·m^2 to 2.5 kg·m^2
- What is Angelina's angular velocity after catching the exercise ball? (Neglect any external torque from the ball)

$$L_o = L_f \Rightarrow I_o \omega_o = I_f \omega_f$$

$$\omega_f = \frac{I_o \omega_o}{I_f} = \frac{(2\ \text{kg}\cdot\text{m}^2)(8\ \text{rad/s})}{(2.5\ \text{kg}\cdot\text{m}^2)} = \boxed{6.4\ \frac{\text{rad}}{\text{s}}}$$

2005 Free Response Question 3



Before Collision After Collision

TOP VIEWS

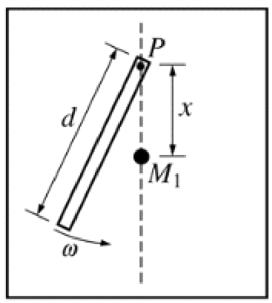
Mech. 3.

A system consists of a ball of mass M_2 and a uniform rod of mass M_1 and length d. The rod is attached to a horizontal frictionless table by a pivot at point P and initially rotates at an angular speed ω, as shown above left. The rotational inertia of the rod about point P is $\frac{1}{3}M_1d^2$. The rod strikes the ball, which is initially at rest. As a result of this collision, the rod is stopped and the ball moves in the direction shown above right. Express all answers in terms of M_1, M_2, ω, d, and fundamental constants.

(a) Derive an expression for the angular momentum of the rod about point P before the collision.

(b) Derive an expression for the speed v of the ball after the collision.

(c) Assuming that this collision is elastic, calculate the numerical value of the ratio M_1/M_2.



Before Collision

(d) A new ball with the same mass M_1 as the rod is now placed a distance x from the pivot, as shown above. Again assuming the collision is elastic, for what value of x will the rod stop moving after hitting the ball?

a) $L_{R_P} = I\omega = \boxed{\frac{M_1 d^2}{3}\omega}$

b) $L_{B_P} = L_{R_P} \Rightarrow \frac{M_1 d^2}{3}\omega = M_2 v d \Rightarrow v = \frac{M_1 d^2 \omega}{3 d M_2} \Rightarrow$

$$\boxed{v = \frac{M_1 d\omega}{3M_2}}$$

c) $K_i = K_f \Rightarrow K_{R_B} = K_{B_A} \Rightarrow \frac{1}{2} I\omega^2 = \frac{1}{2} M_2 v^2 \Rightarrow$

$$\frac{\cancel{M_1}\cancel{d^2}\cancel{\omega^2}}{\cancel{2\cdot 3}\,6} = \frac{\cancel{M_2} M_1^2 \cancel{d^2}\cancel{\omega^2}}{18\,\cancel{2\cdot 9}\, M_2^2} \Rightarrow \frac{1}{6} = \frac{M_1}{18 M_2} \Rightarrow \boxed{\frac{M_1}{M_2} = 3}$$

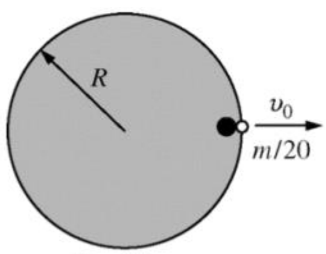
d) $L_{B_p} = L_{R_p} \Rightarrow \frac{M_1 d^2}{3}\omega = M_1 v x \Rightarrow v = \frac{d^2}{3x}\omega$

$$K_{R_B} = K_{B_A} \Rightarrow \frac{1}{2} I\omega^2 = \frac{1}{2} M_1 v^2 \Rightarrow$$

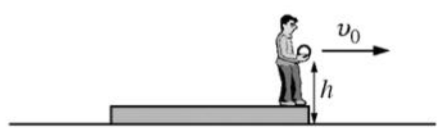
$$\cancel{\tfrac{1}{2}}\left(\frac{M_1 d^2}{3}\right)\omega^2 = \cancel{\tfrac{1}{2}} M_1 \frac{d^4}{9x^2}\omega^2 \Rightarrow \frac{\cancel{M_1}\cancel{d^2}\cancel{\omega^2}}{1\,\cancel{3}} = \frac{\cancel{M_1} d^{\cancel{4}\,2}\cancel{\omega^2}}{3\,\cancel{9}\,x^2} \Rightarrow$$

$$1 = \frac{d^2}{3x^2} \Rightarrow x^2 = \frac{d^2}{3} \Rightarrow \boxed{x = \frac{d}{\sqrt{3}}}$$

2014 Free Response Question 3



Top View



Side View

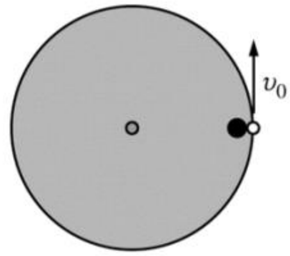
Mech. 3.

A large circular disk of mass m and radius R is initially stationary on a horizontal icy surface. A person of mass $m/2$ stands on the edge of the disk. Without slipping on the disk, the person throws a large stone of mass $m/20$ horizontally at initial speed v_0 from a height h above the ice in a radial direction, as shown in the figures above. The coefficient of friction between the disk and the ice is μ. All velocities are measured relative to the ground. The time it takes to throw the stone is negligible. Express all algebraic answers in terms of m, R, v_0, h, μ, and fundamental constants, as appropriate.

(a) Derive an expression for the length of time it will take the stone to strike the ice.

(b) Assuming that the disk is free to slide on the ice, derive an expression for the speed of the disk and person immediately after the stone is thrown.

(c) Derive an expression for the time it will take the disk to stop sliding.



Top View

The person now stands on a similar disk of mass m and radius R that has a fixed pole through its center so that it can only rotate on the ice. The person throws the same stone horizontally in a tangential direction at initial speed v_0, as shown in the figure above. The rotational inertia of the disk is $mR^2/2$.

(d) Derive an expression for the angular speed ω of the disk immediately after the stone is thrown.

(e) The person now stands on the disk at rest $R/2$ from the center of the disk. The person now throws the stone horizontally with a speed v_0 in the same direction as in part (d). Is the angular speed of the disk immediately after throwing the stone from this new position greater than, less than, or equal to the angular speed found in part (d) ?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

a) Vert
$\downarrow +y$
$V_0 = 0$
$\Delta y = h$
$a_y = g$

$$\Delta y = \cancel{V_0}^{\,0} t + \frac{1}{2} a_y t^2 \Rightarrow h = \frac{1}{2} g t^2 \Rightarrow \boxed{t = \sqrt{\frac{2h}{g}}}$$

b) $$P_i = P_f \Rightarrow \left(m + \frac{m}{2}\right) v + \frac{m}{20} V_0 = 0 \Rightarrow \frac{3m}{2} v + \frac{m}{20} V_0 = 0 \Rightarrow$$

$$30mv + mV_0 = 0 \Rightarrow 30v + V_0 = 0 \Rightarrow \boxed{V = \frac{-V_0}{30}}$$

e) $$I_{TOT} = \frac{1}{2} mR^2 + \frac{m}{2}\left(\frac{R}{2}\right)^2 = \frac{1}{2} mR^2 + \frac{mR^2}{8} = \frac{5mR^2}{8}$$ $I \downarrow$

$$L_i = L_f \Rightarrow I\omega_i = I\omega_f \Rightarrow 0 = \frac{m}{20} V_0 \frac{R}{2} + \frac{5mR^2}{8}\omega \Rightarrow$$

$$\frac{mV_0}{40} = \frac{5mR}{8}\omega \Rightarrow \omega = \frac{8V_0}{200R} = \frac{V_0}{25R}$$ Less Than

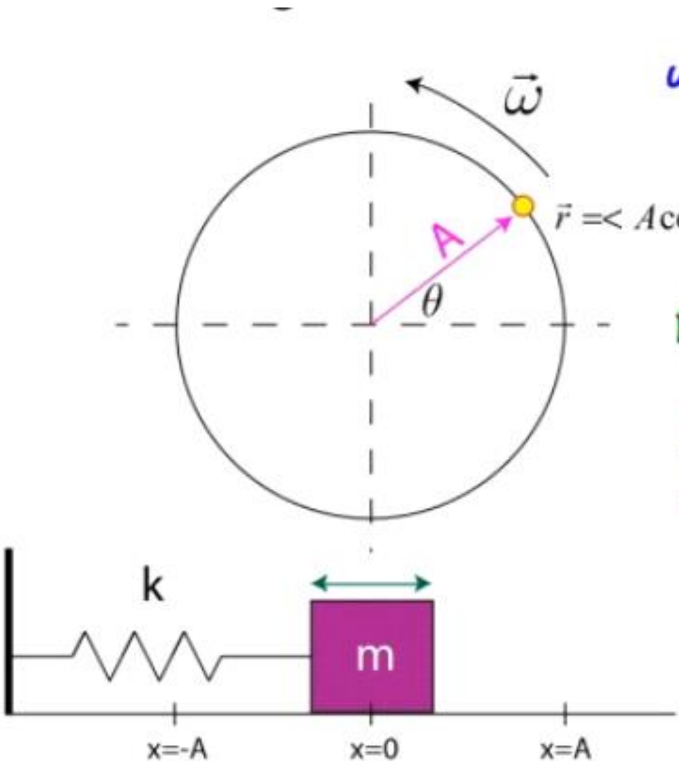
8.1 - Oscillations

Thursday, March 16, 2017 11:32 PM

Simple Harmonic Motion

- Simple harmonic motion (SHM) is motion in which a restoring force is directly proportional to the displacement of an object
- Nature's response to a perturbation or disturbance is often SHM

Circular Motion vs. SHM



$$\omega = \frac{d\theta}{dt} \Rightarrow \int_{t=0}^{t} \omega\, dt = \int_{\theta=0}^{\theta} d\theta \Rightarrow$$

$$\omega t = \theta$$

$$F = ma = -kx \xrightarrow{a = \frac{d^2x}{dt^2}} m\frac{d^2x}{dt^2} = -kx \Rightarrow$$

$$\frac{d^2x}{dt^2} + \underbrace{\frac{k}{m}}_{\omega^2} x = 0$$

$$x(t) = A\cos(\omega t)$$

$$\omega = \sqrt{\frac{k}{m}}$$

Position, Velocity, Acceleration

$$\theta = \omega t$$

$$x(t) = A\cos(\omega t)$$

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(A\cos(\omega t)\right) = -\omega A \sin(\omega t) \qquad v_{max} = \omega A$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(-\omega A \sin \omega t\right) = -\omega^2 A\cos(\omega t) \qquad a_{max} = \omega^2 A$$

Frequency and Period

- Frequency
  - Frequency is the number of revolutions or cycles which occur each second
  - Symbol is f
  - Units are 1/s, or Hertz (Hz)
  - $f = \frac{number\ of\ cycles}{second} = \frac{number\ of\ revolution}{second}$
- Period
  - Period is the time it takes for one complete revolution, or cycle.
  - Symbol is T
  - Unites are seconds (s)
  - T = time for 1 cycle = time for 1 revolution
- Relationship
  - $f = \frac{1}{T}$
  - $T = \frac{1}{f}$

Angular Frequency

- Angular frequency is the number of radians per second, and it corresponds to the angular velocity for an object traveling in uniform circular motion
- Relationship
  - $\omega = 2\pi f = \frac{2\pi}{T}$
  - $T = 2\pi\omega = \frac{1}{f}$
  - $f = \frac{\omega}{2\pi} = \frac{1}{T}$

Example 1: Oscillating System

- An oscillating system is created by a releasing an object from a maximum displacement of 0.2 meters. The object makes 60 complete oscillations in one minute
- Determine the object's angular frequency

$$\omega = 2\pi f = 2\pi(1\ Hz) = 2\pi\ \frac{rad}{s}$$

- What is the object's position at time t=10s?

$$x = A\cos(\omega t) \xrightarrow[A=.2m]{\omega=2\pi} x = 0.2\cos(2\pi t) \xrightarrow{t=10s}$$

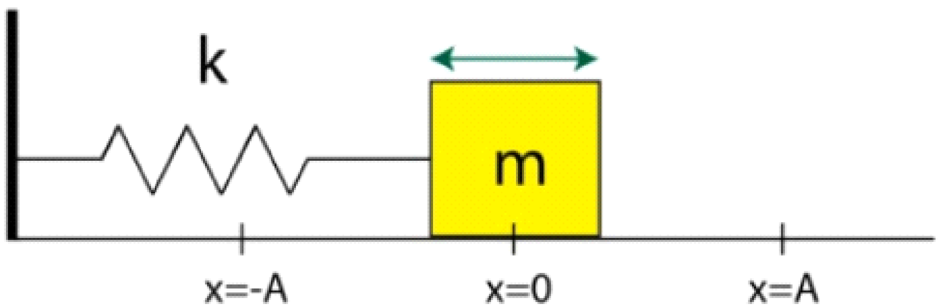
$$x = 0.2\cos(2\pi\cdot 10) = \boxed{0.2\,m}$$

- At what time is the object at x=0.1m?

$$x = A\cos(2\pi t) \Rightarrow \cos(2\pi t) = \frac{x}{A} \Rightarrow 2\pi t = \cos^{-1}\left(\frac{x}{A}\right) \Rightarrow$$

$$t = \frac{\cos^{-1}(x/A)}{2\pi} = \frac{\cos^{-1}(.1/.2)}{2\pi} = \boxed{0.167\,s}$$

Mass on a Spring



$$T_s = 2\pi\sqrt{\frac{m}{k}}$$

$$f_s = \frac{1}{T_s} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

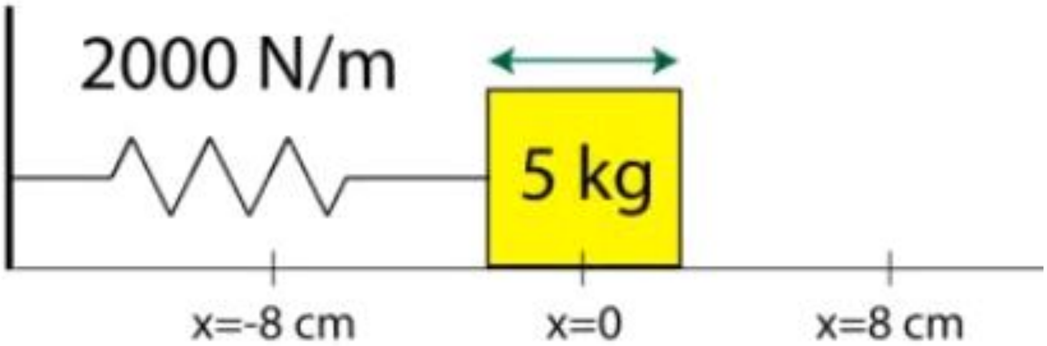
$$2\pi f = \sqrt{\frac{k}{m}}$$

$\omega \rightarrow$ angular freq $= 2\pi f$

$$\therefore\ \omega = \sqrt{\frac{k}{m}}$$

Example 2: Analysis of Spring-Block System

- A 5-kg block is attached to a 2000 N/m spring as shown and displaced a distance of 8 cm from its equilibrium position before being released.
- Determine the period of oscillation, the frequency, and the angular frequency for the block



$$T_s = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{5}{2000}} = \boxed{0.314\,s}$$

$$f = \frac{1}{T} = \frac{1}{.314\,s} = \boxed{3.18\ Hz}$$

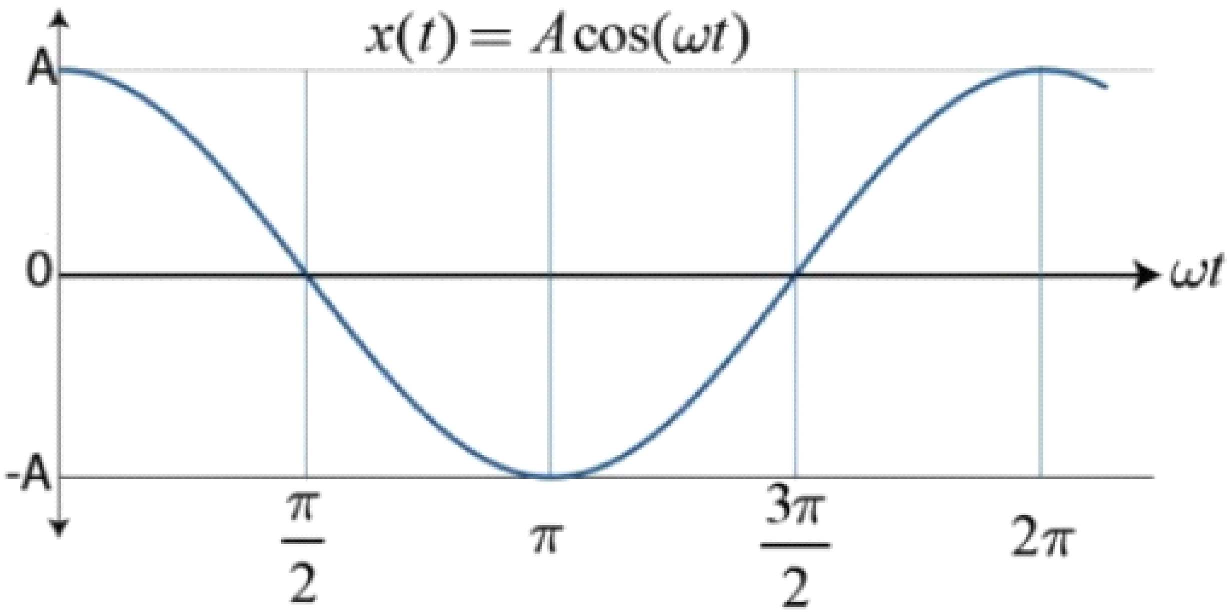
$$\omega = 2\pi f = 2\pi(3.18\ Hz) = \boxed{20\ \frac{rad}{s}}$$

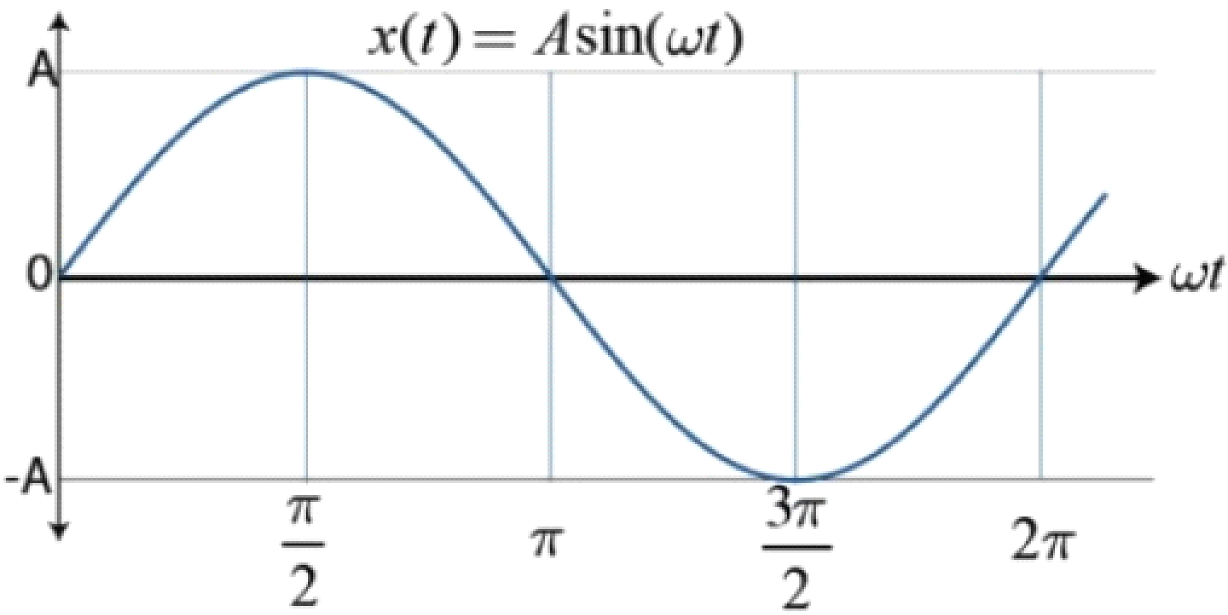
General Form of SHM

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$x(t) = A\cos(\omega t + \phi)$$

Graphing SHM





Energy of SHM

- When an object undergoes SHM, kinetic and potential energy both vary with time, although total energy (E=K+U) remains constant

$$W=\int_x^0 \vec{F_x}\cdot d\vec{x} = \int_x^0 -kx\,dx = -k\frac{x^2}{2}\Big|_x^0 = \frac{1}{2}kx^2 = U_s$$

$$x(t)=A\cos(\omega t) \Rightarrow U_s = \frac{1}{2}k\left(A\cos(\omega t)\right)^2 = \frac{1}{2}kA^2\cos^2(\omega t)$$

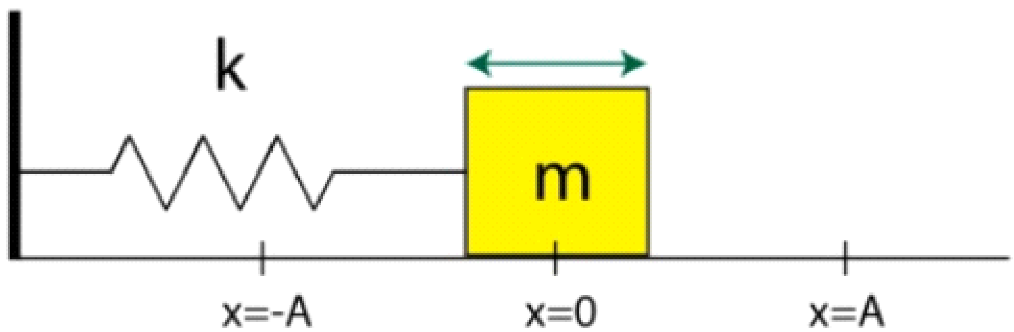
$$v(t)=\frac{dx}{dt} = -\omega A\sin(\omega t) \Rightarrow K=\frac{1}{2}mv^2 = \frac{1}{2}m\left[-\omega A\sin(\omega t)\right]^2 \Rightarrow$$

$$K=\frac{1}{2}m\omega^2A^2\sin^2(\omega t) \xrightarrow{\omega^2=\frac{k}{m}} K=\frac{1}{2}kA^2\sin^2(\omega t)$$

$$E=K+U=\frac{1}{2}kA^2\sin^2(\omega t)+\frac{1}{2}kA^2\cos^2(\omega t)$$

$$E=\frac{1}{2}kA^2$$

Horizontal Spring Oscillator

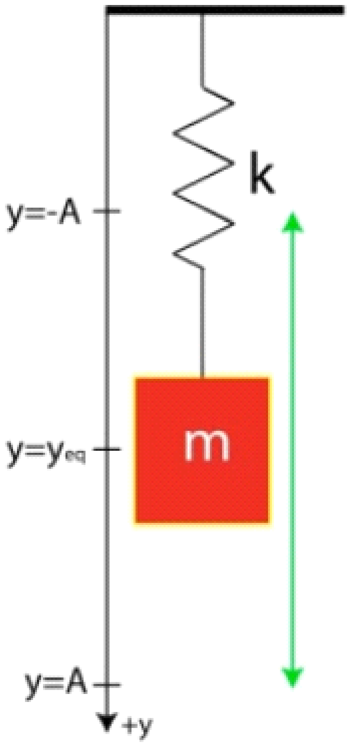


$$F=ma=-kx \xRightarrow{a=\frac{d^2x}{dt^2}} m\frac{d^2x}{dt^2}=-kx \Rightarrow \frac{d^2x}{dt^2}+\overbrace{\frac{k}{m}}^{\omega^2}x^2=0$$

$$x(t)=A\cos(\omega t+\phi) \qquad \omega=\sqrt{\frac{k}{m}}$$

$$T=\frac{2\pi}{\omega}=\frac{2\pi}{\sqrt{\frac{k}{m}}}=\boxed{2\pi\sqrt{\frac{m}{k}}}$$

Vertical Spring Oscillator



$$F_{NET_y} = mg - ky = ma_y \Rightarrow mg - ky_{eq} = 0$$

$$\Rightarrow y_{eq} = \frac{mg}{k}$$

$F_s = ky$

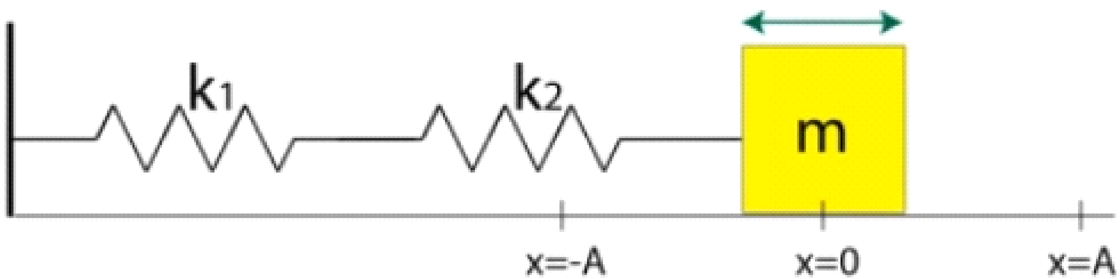
mg

$$F_{NET_y} = mg - k(y_{eq} + A) \Rightarrow F_{NET} =$$

$$\underbrace{(mg - ky_{eq})}_{0} - kA$$

$$F_{NET_y} = -kA$$

Springs in Series



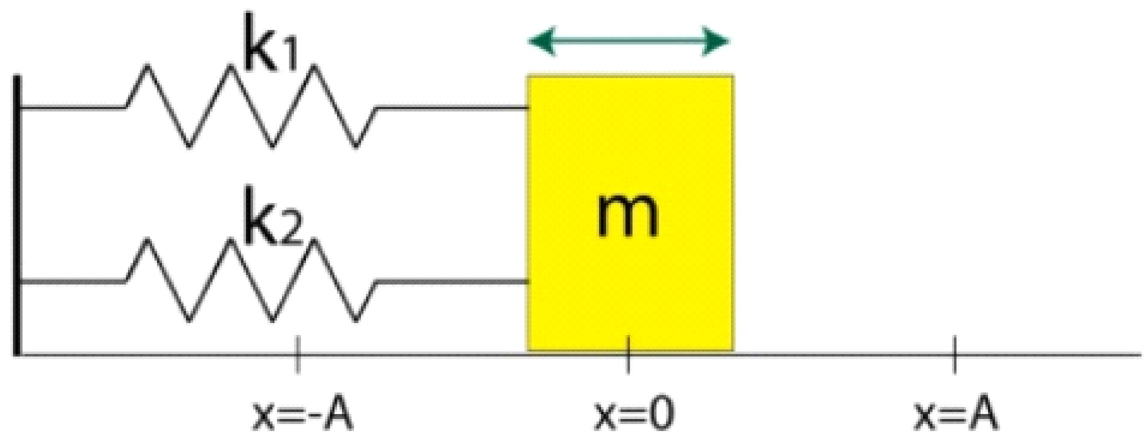
$$F = -k_1 x_1 = -k_2 x_2 \Rightarrow x_1 = \frac{k_2}{k_1} x_2$$

$$F = -k_{eq}(x_1 + x_2) \Rightarrow F = -k_{eq}\left(\frac{k_2}{k_1} x_2 + x_2\right) \xrightarrow{F = -k_2 x_2}$$

$$-k_2 x_2 = -k_{eq} x_2 \left(\frac{k_2}{k_1} + 1\right) \Rightarrow k_2 = k_{eq}\left(\frac{k_2}{k_1} + 1\right) \Rightarrow$$

$$\boxed{\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}}$$

Springs in Parallel



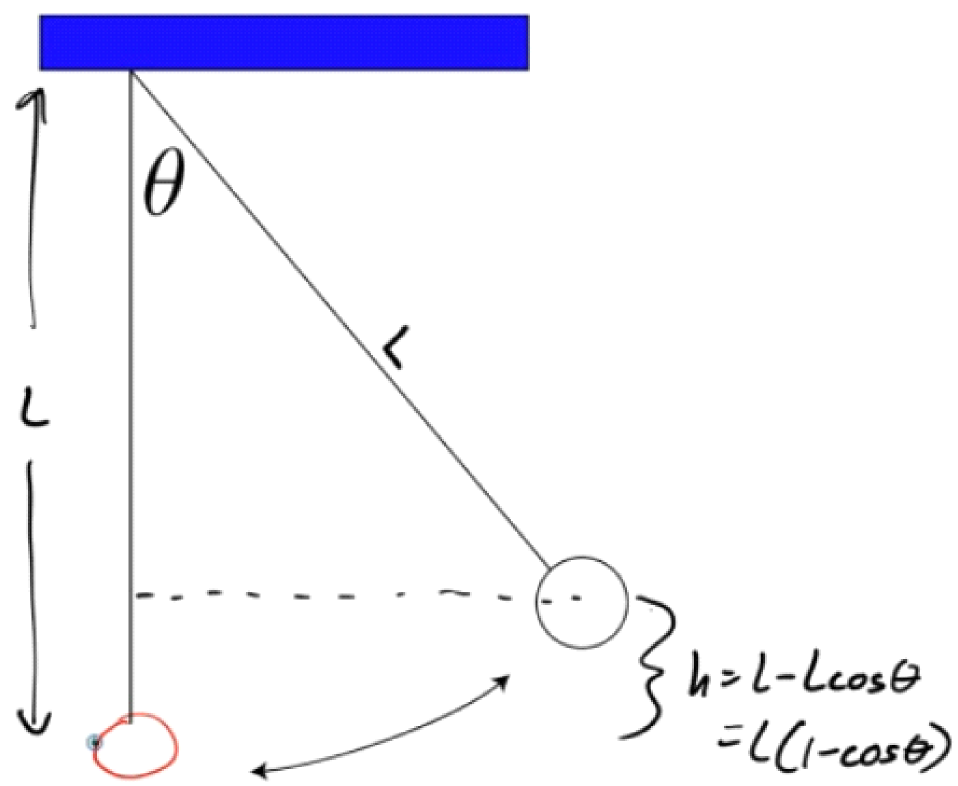
$$F = k_1 x + k_2 x = (k_1 + k_2)x$$

$$(k_1 + k_2)x = k_{eq}\, x$$

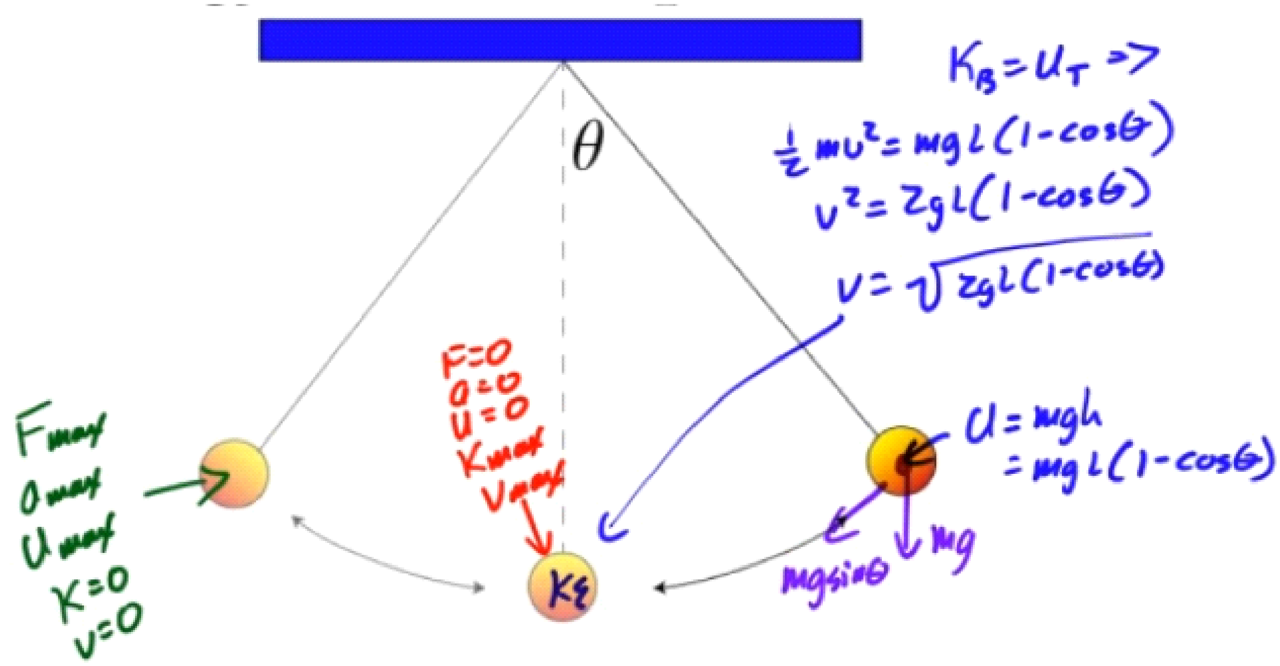
$$K_{eq} = k_1 + k_2$$

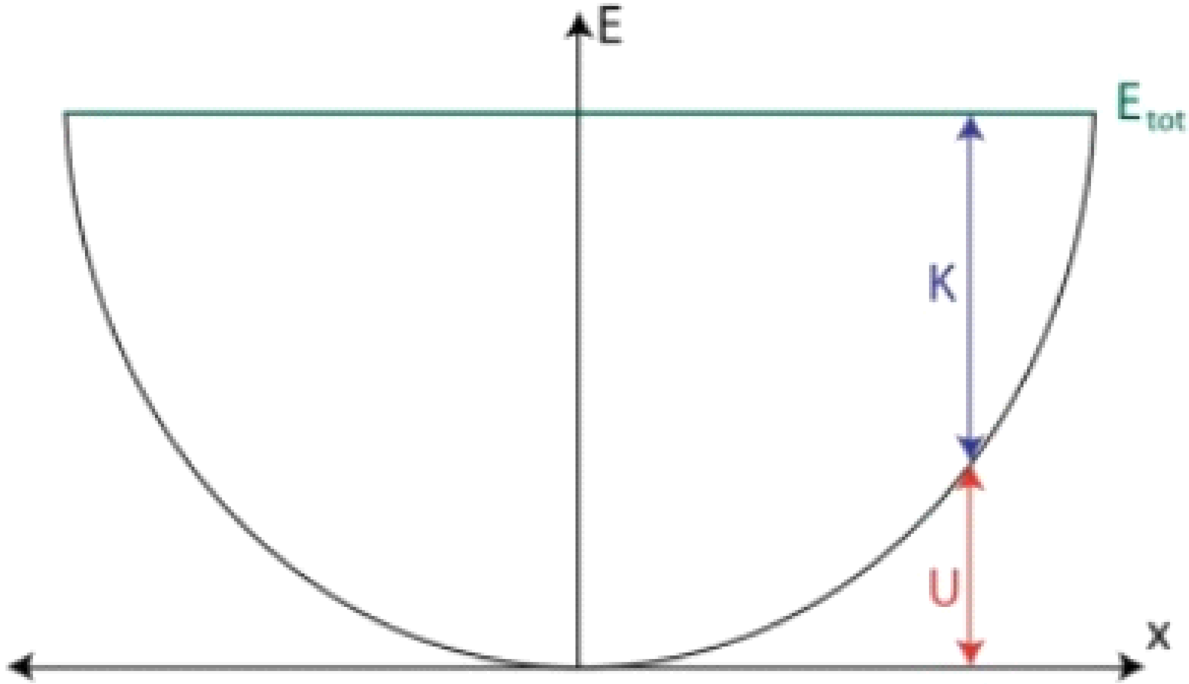
The Pendulum

- A mass m is attached to a light string that swings without friction about the vertical equilibrium position

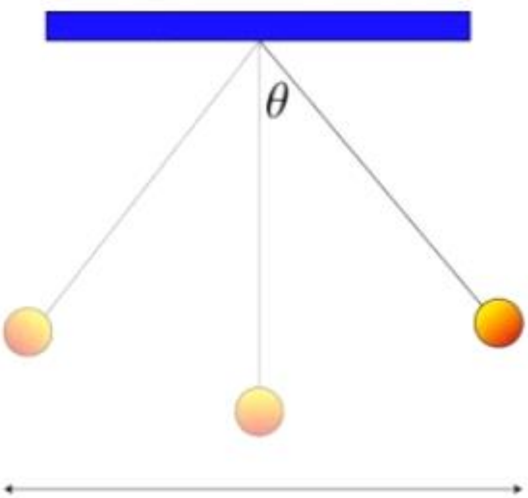


Energy and the Simple Pendulum



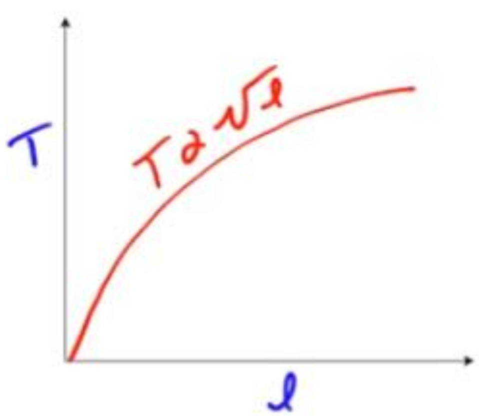


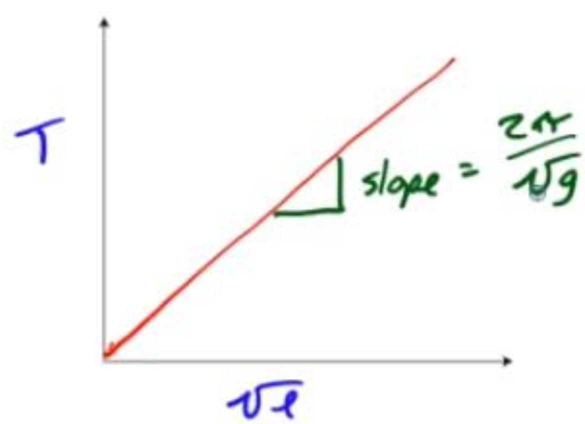
Frequency and Period of a Pendulum



$$T = 2\pi\sqrt{\frac{l}{g}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

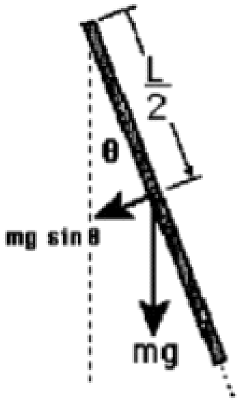
Note: Assume θ is small due to small angle approximation.





Period of a Physical Pendulum

The Physical Pendulum



A physical pendulum is an oscillating body that rotates according to the location of its center of mass rather than a simple pendulum where all the mass is located at the end of a light string.

$$Fr\sin\theta = \tau = I\alpha$$

$$-mg\sin\theta d = I\alpha, \quad d = L/2$$

$$-mgd = I\alpha \quad if\theta <<<, \sin\theta = \theta$$

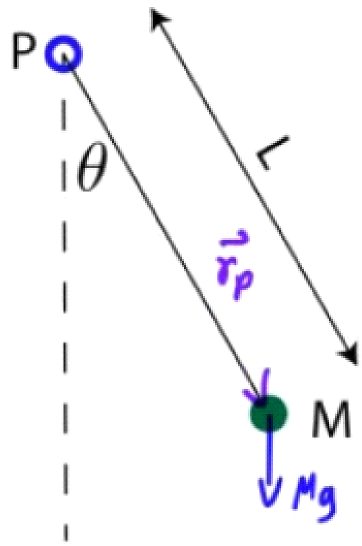
$$\alpha + \left(\frac{mgd}{I}\right)\theta = 0$$

$$\omega = \sqrt{\frac{mgd}{I}}, \quad \omega = \frac{2\pi}{T}$$

$$T_{physical\ pendulum} = 2\pi\sqrt{\frac{I}{mgd}}$$

It is important to understand that "d" is the lever arm distance or the distance from the COM position to the point of rotation. It is also the same "d" in the Parallel Axes theorem.

Example 3: Deriving Period of a Simple Pendulum



$$\vec{\tau}_p = \vec{r}_p \times \vec{F} = \vec{r}_p \times M\vec{g} \Rightarrow |\vec{\tau}_p| = Mgl\sin\theta \Rightarrow$$

$$-Mgl\sin\theta = I_p\alpha \xrightarrow{\sin\theta = \theta} -Mgl\theta = I_p\alpha$$

$$\xrightarrow{\alpha = \frac{d^2\theta}{dt^2}} -Mgl\theta = I_p\frac{d^2\theta}{dt^2} \Rightarrow$$

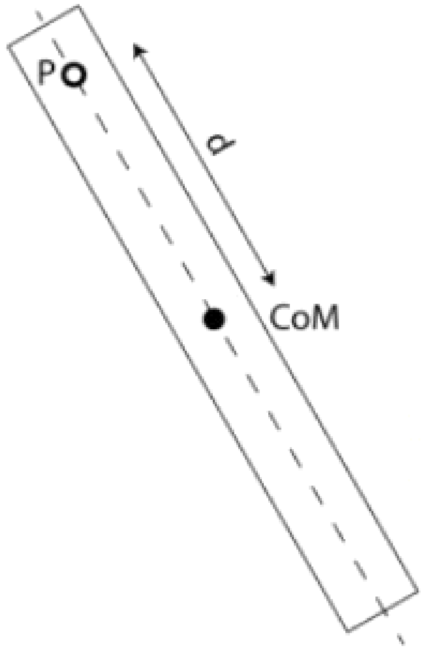
$$\frac{d^2\theta}{dt^2} + \overbrace{\left(\frac{Mgl}{I_p}\right)}^{\omega^2}\theta = 0$$

$$\theta = A\cos(\omega t) \qquad I_p = ML^2$$

$$\omega^2 = \frac{Mgl}{ML^2} = \frac{g}{L} \Rightarrow \omega = \sqrt{\frac{g}{L}}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/l}} = \boxed{2\pi\sqrt{\frac{L}{g}}}$$

Example 4: Deriving Period of a Physical Pendulum



$$\vec{\tau}_{Net} = \vec{r} \times \vec{F} \qquad \theta = A\cos(\omega t)$$

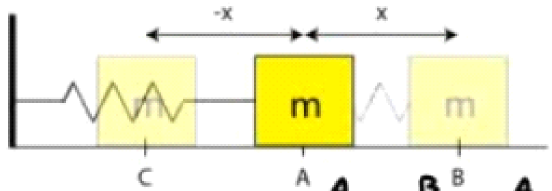
$$\omega^2 = \frac{Mgd}{I_P} \Rightarrow \omega = \sqrt{\frac{Mgd}{I_P}}$$

$$I_P = I_{cm} + Md^2 = \frac{ML^2}{12} + Md^2$$

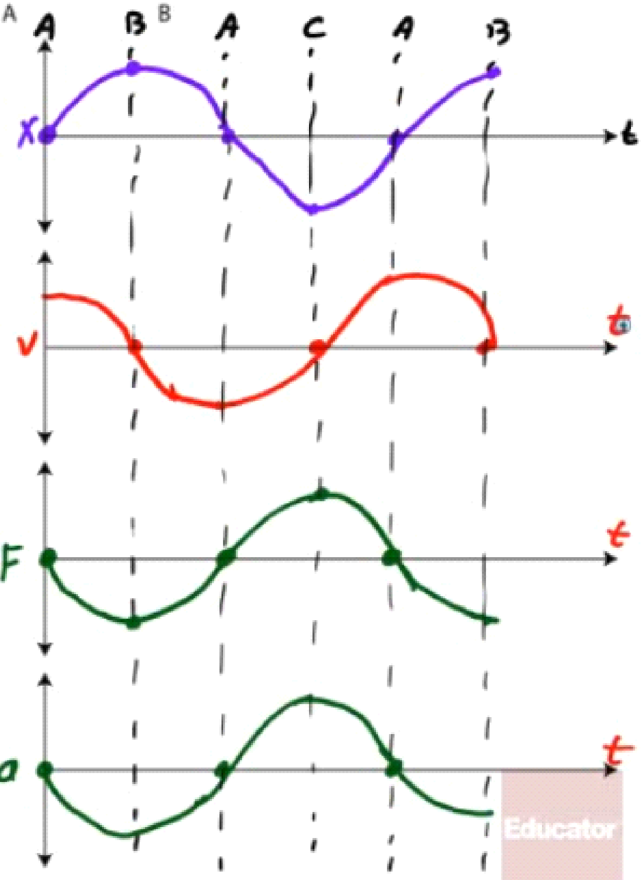
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\frac{ML^2}{12} + Md^2}{Mgd}} \Rightarrow$$

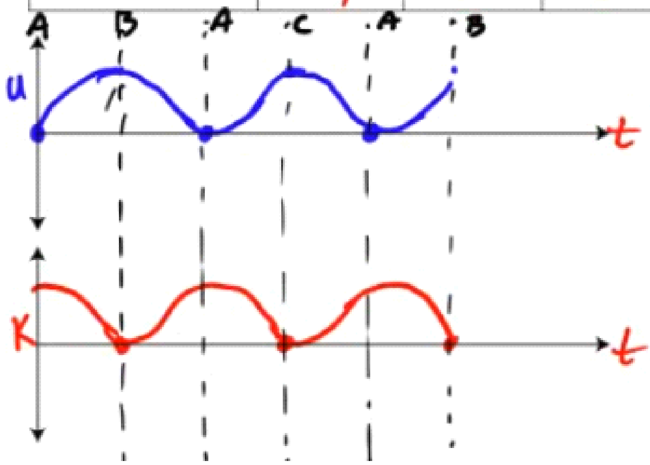
$$T = 2\pi\sqrt{\frac{\frac{L^2}{12} + d^2}{gd}} = \boxed{2\pi\sqrt{\frac{L^2 + 12d^2}{12gd}}}$$

Example 5: Summary of Spring-Block System



	A	B	C
Displacement	0	x	-x
Velocity	max	0	0
Force	0	-max	max
Acceleration	0	-max	max
U	0	max	max
K	max	0	0





Example 6: Harmonic Oscillator Analysis

- A 2-kg block is attacked to a spring. A force of 20 N stretches the spring to a displacement of 0.5 meter
- The spring constant

$$F = kx \Rightarrow k = \frac{F}{x} = \frac{20\,N}{.5\,m} = \boxed{40\,\frac{N}{m}}$$

- The total energy

$$E_T = U_s = \frac{1}{2}kx^2 = \frac{1}{2}\left(40\,\frac{N}{m}\right)(.5\,m)^2 = \boxed{5\,J}$$

- The speed at the equilibrium position

$$U_s = K = \frac{1}{2}mv^2 = 5\,J \Rightarrow v = \sqrt{\frac{2(5)}{2}} = \boxed{2.24\,\frac{m}{s}}$$

- The speed at x=0.30 meters

$$\mathcal{E}_T = U_s + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \Rightarrow mv^2 = 2\mathcal{E}_T - kx^2 \Rightarrow v = \sqrt{\frac{2\mathcal{E}_T - kx^2}{m}} \Rightarrow$$

$$v = \sqrt{\frac{2(5) - 40(.3)^2}{2}} = \boxed{1.79\ \frac{m}{s}}$$

- The speed at x=-0.4 meters

$$v = \sqrt{\frac{2\mathcal{E}_T - kx^2}{m}} = \sqrt{\frac{2(5) - 40(-.4)^2}{2}} = \boxed{1.34\ \frac{m}{s}}$$

- The acceleration at the equilibrium position

$$@\ x = 0,\ F = 0\ \therefore \boxed{a = 0}$$

- The magnitude of the acceleration at x=0.5 meters.

$$F = -kx = (-40)(.5) = -20N$$

$$a = \frac{F}{m} = \frac{-20N}{2\ kg} = \boxed{-10\ \frac{m}{s^2}}$$

- The net force at equilibrium position

$$@\ \text{equil},\ a = 0,\ F = 0$$

- The net force at x=0.25 meter

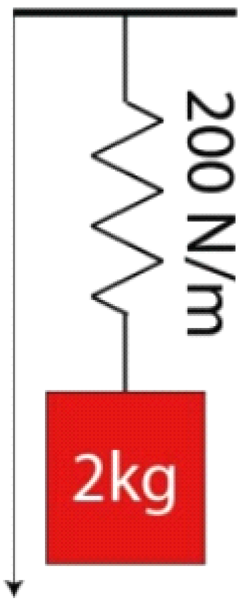
$$F = kx = (-40)(.25) = -10N$$

- Where does kinetic energy = potential energy

$$U_s = 2.5J = \frac{1}{2}kx^2 \Rightarrow x^2 = \frac{2(2.5J)}{40} = .125\ m^2 \Rightarrow$$

$$x = \sqrt{.125} = 0.354m$$

Example 7: Vertical Spring Block Oscillator



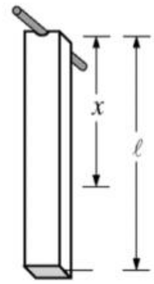
- A 2-kg block attached to an un-stretched spring of spring constant k=200 N/m as shown in the diagram below is released from rest. Determine the period of the block's oscillation and the maximum displacement of the block from its equilibrium while undergoing simple harmonic motion.

$$T_s = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{200}} = \boxed{0.63\,s}$$

$$U_g = U_s \Rightarrow mg\Delta y = \frac{1}{2}k\Delta y^2 \Rightarrow \Delta y = \frac{2mg}{k}$$

$$A = \frac{\Delta y}{2} = \frac{2mg}{2k} = \frac{mg}{k} = \frac{(2)(9.8)}{200} = \boxed{0.1\,m}$$

2009 Free Response Question 2



Mech. 2.

You are given a long, thin, rectangular bar of known mass M and length ℓ with a pivot attached to one end. The bar has a nonuniform mass density, and the center of mass is located a known distance x from the end with the pivot. You are to determine the rotational inertia I_b of the bar about the pivot by suspending the bar from the pivot, as shown above, and allowing it to swing. Express all algebraic answers in terms of I_b, the given quantities, and fundamental constants.

(a)

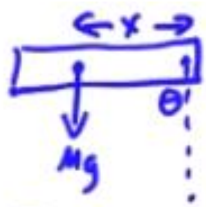
i. By applying the appropriate equation of motion to the bar, write the differential equation for the angle θ the bar makes with the vertical.

ii. By applying the small-angle approximation to your differential equation, calculate the period of the bar's motion.

(b) Describe the experimental procedure you would use to make the additional measurements needed to determine I_b. Include how you would use your measurements to obtain I_b and how you would minimize experimental error.

(c) Now suppose that you were not given the location of the center of mass of the bar. Describe an experimental procedure that you could use to determine it, including the equipment that you would need.

a) $\tau_{NET} = I\alpha$



$$-Mgx\sin\theta = I_b\,\alpha$$

$$-Mgx\sin\theta = I_b\,\frac{d^2\theta}{dt^2}$$

b) for small θ, $\sin\theta \approx \theta$

$$-Mgx\,\theta = I_b\,\frac{d^2\theta}{dt^2} \Rightarrow$$

$$\frac{d^2\theta}{dt^2} + \underbrace{\frac{Mgx}{I_b}}_{\omega^2}\theta = 0$$

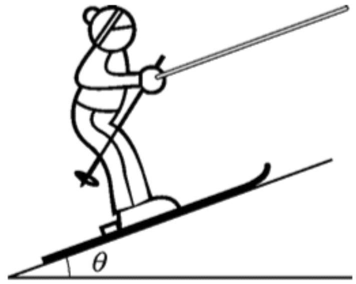
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{Mgx}{I_b}}} = 2\pi\sqrt{\frac{I_b}{Mgx}}$$

b)

$$T = 2\pi\sqrt{\frac{I_b}{Mgx}} \Rightarrow T^2 = 4\pi^2\,\frac{I_b}{Mgx} \Rightarrow \boxed{I_b = \frac{T^2 Mgx}{4\pi^2}}$$

c) Balance

2010 Free Response Question 3



Mech. 3.

A skier of mass m will be pulled up a hill by a rope, as shown above. The magnitude of the acceleration of the skier as a function of time t can be modeled by the equations

$$a = a_{max} \sin\frac{\pi t}{T} \qquad (0 < t < T)$$
$$= 0 \qquad (t \geq T),$$

where a_{max} and T are constants. The hill is inclined at an angle θ above the horizontal, and friction between the skis and the snow is negligible. Express your answers in terms of given quantities and fundamental constants.

(a) Derive an expression for the velocity of the skier as a function of time during the acceleration. Assume the skier starts from rest.

(b) Derive an expression for the work done by the net force on the skier from rest until terminal speed is reached.

(c) Determine the magnitude of the force exerted by the rope on the skier at terminal speed.

(d) Derive an expression for the total impulse imparted to the skier during the acceleration.

(e) Suppose that the magnitude of the acceleration is instead modeled as $a = a_{max}e^{-\pi t/2T}$ for all $t > 0$, where a_{max} and T are the same as in the original model. On the axes below, sketch the graphs of the force exerted by the rope on the skier for the two models, from $t = 0$ to a time $t > T$. Label the original model F_1 and the new model F_2.

a) $v = \int_0^t a(t)dt = \int_0^t a_{max} \sin\frac{\pi t}{T}dt = a_{max}\frac{T}{\pi}\int \sin\frac{\pi t}{T}dt\frac{\pi}{T} \Rightarrow$

$v = -a_{max}\frac{T}{\pi}\cos\left(\frac{\pi t}{T}\right)\Big|_0^t = -\frac{a_{max}T}{\pi}\left(\cos\frac{\pi t}{T} - 1\right) \Rightarrow$

$\boxed{v = \frac{a_{max}T}{\pi}\left(1 - \cos\frac{\pi t}{T}\right)}$

b) $W = \Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \cancel{\frac{1}{2}mv_i^2}^{0} \Rightarrow$

$v_f = v(T) = \frac{a_{max}T}{\pi}(1 - -1)$

$= \frac{2a_{max}T}{\pi}$

$W = \frac{1}{2}m\left(\frac{4a_{max}^2T^2}{\pi^2}\right) = \boxed{\frac{2ma_{max}^2T^2}{\pi^2}}$

c) 

$$F_{NET_x} = 0 = F_T - mg\sin\theta \Rightarrow \boxed{F_T = mg\sin\theta}$$

d) $$J = \int F\,dt = \int ma\,dt = \int_{t=0}^{T} m a_{max} \sin\frac{\pi t}{T}\,dt =$$

$$m a_{max}\frac{T}{\pi}\int \sin\frac{\pi t}{T}\,dt\left(\frac{\pi}{T}\right) = \frac{-m a_{max} T}{\pi}\cos\frac{\pi t}{T}\Big|_0^T =$$

$$-\frac{m a_{max} T}{\pi}\left(\cos\pi - 1\right) = \boxed{\frac{2 m a_{max} T}{\pi}}$$

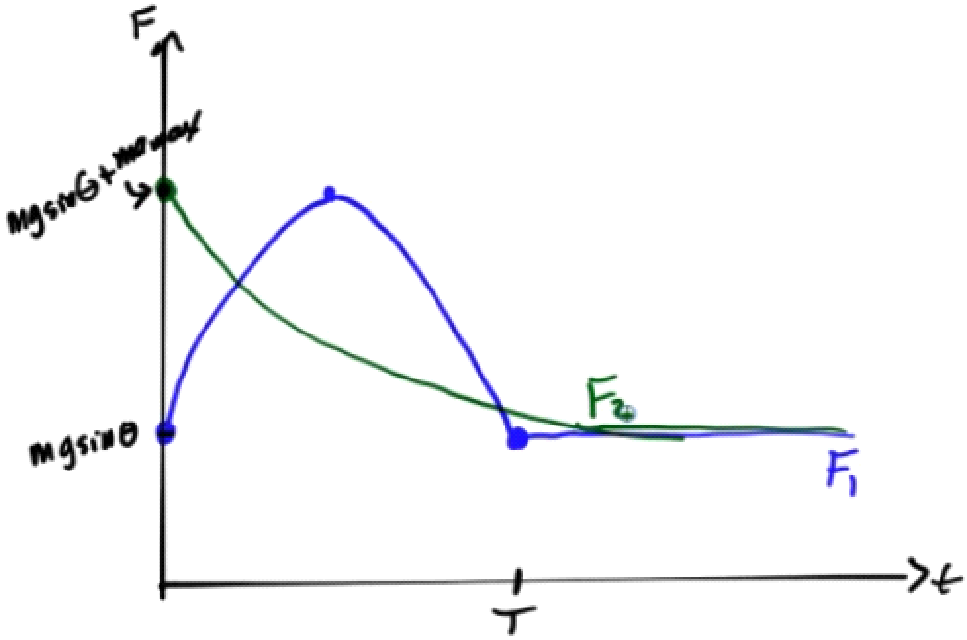
e)

F_1 ORIGINAL

$$F_T - mg\sin\theta = ma_{max}\sin\frac{\pi t}{T} \Rightarrow$$

$$F_T = mg\sin\theta + ma_{max}\sin\frac{\pi t}{T}$$

$F2$: $F_T - mg\sin\theta = ma_{max}e^{-\frac{\pi t}{2T}} \Rightarrow F_T = mg\sin\theta + ma_{max}e^{-\frac{\pi t}{2T}}$



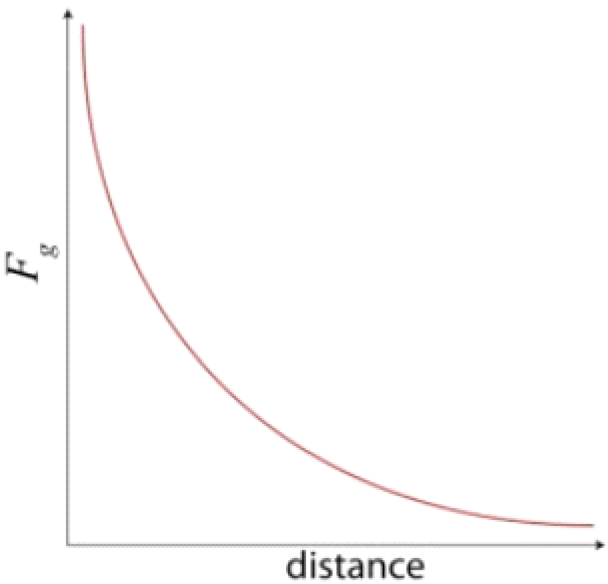
9.1 - Gravity & Orbits

Thursday, March 16, 2017 11:32 PM

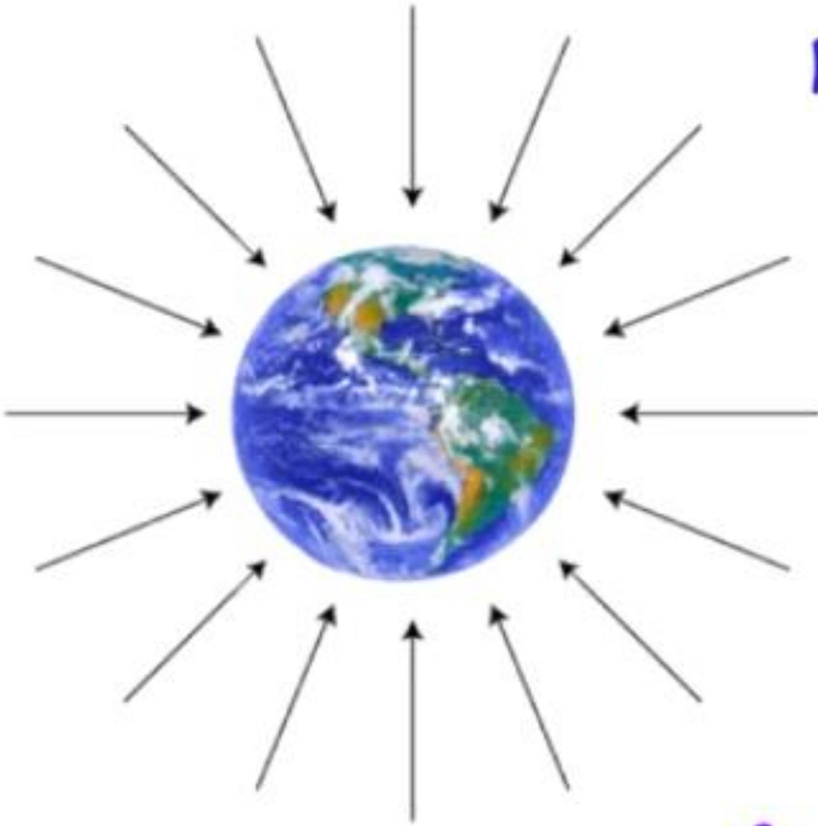
Newton's Law of Universal Gravitation

$$F_g = -\frac{Gm_1m_2}{r^2}\hat{r}$$

$$G = 6.67\times10^{-11}\ {N\bullet m^2}/{kg^2}$$



Gravitational Field Strength



$$F_g = \frac{G m_1 \cancel{m_2}}{r^2} = \cancel{m_2} g$$

$$g = \frac{G m_1}{r^2}$$

$$\left[\frac{m}{s^2}\right] = \left[\frac{N}{kg}\right]$$

$$r_e = 6378 \text{ km}$$

$$g = 9.8 \frac{m}{s^2} = 9.8 \frac{N}{kg}$$

Gravitational Field of a Hollow Shell

- Inside a hollow sphere, the gravitational field is 0. Outside a hollow sphere, you can treat the sphere as if it's entire mass was concentrated at the center, and then calculate the gravitational field

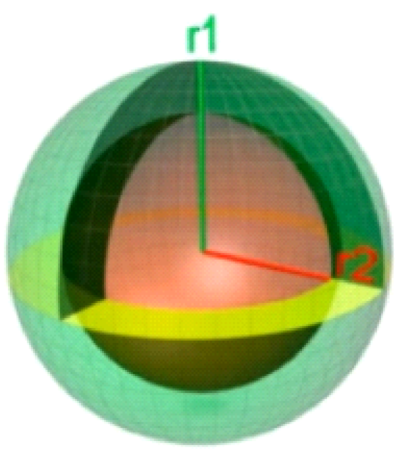
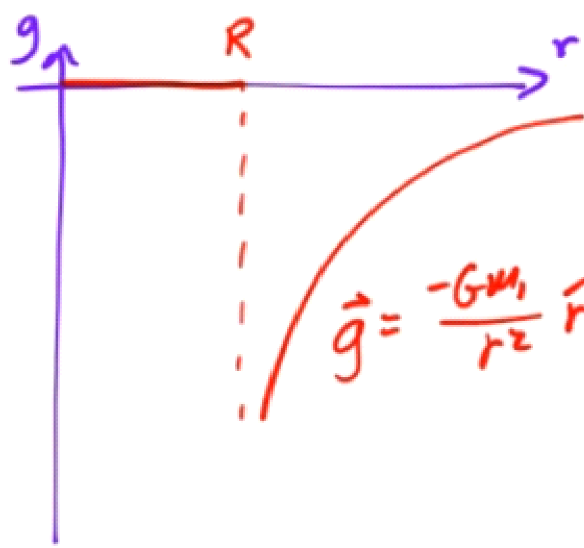


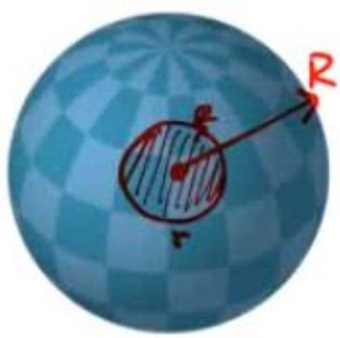
Illustration Courtesy of Dominique Toussaint



Gravitational Field Inside a Solid Shell

- Outside a solid sphere, treat the sphere as if all the mass is at the center of the sphere.

Inside the sphere, treat the sphere as if the mass inside the radius is all at the center. Only the mass inside the "radius of interest" counts

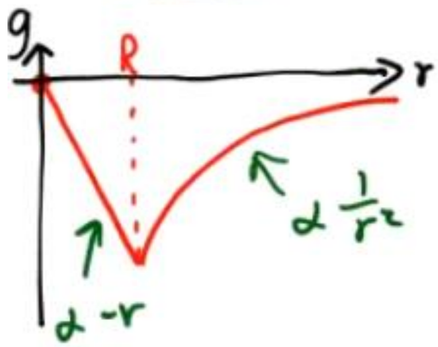


$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$V_{enc} = \frac{4}{3}\pi r^3$$

$$M_{enc} = \rho V_{enc} = \frac{M \cancel{\frac{4}{3}\pi} r^3}{\cancel{\frac{4}{3}\pi} R^3}$$

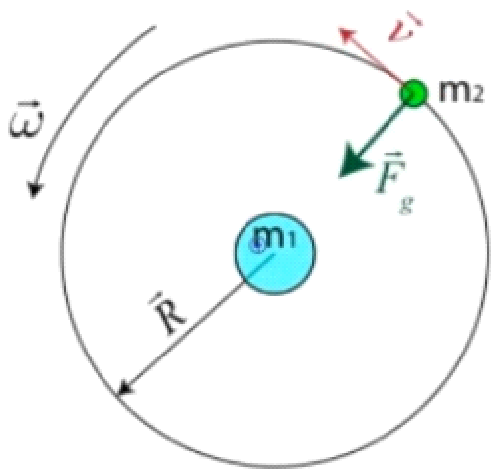
$$M_{enc} = \frac{M r^3}{R^3}$$



$$g = \frac{-GM_{enc}}{r^2} = \frac{-G\frac{Mr^3}{R^3}}{r^2} = -\frac{GM}{R^3} r$$

for $r \le R$

Velocity in Circular Orbit

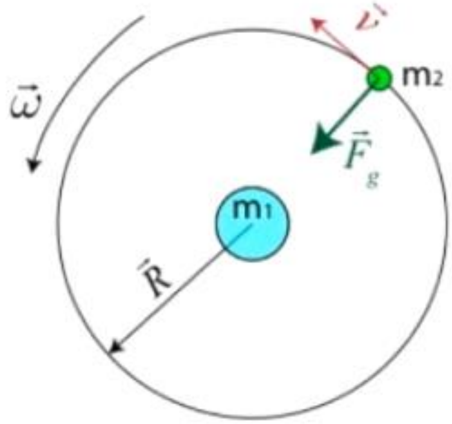


$$F_{NET_c} = m_2 a_c = F_g \qquad F_g = \frac{G m_1 m_2}{R^2} \qquad a_c = \frac{v^2}{R}$$

$$\frac{m_2 v^2}{R} = \frac{G m_1 m_2}{R^2} \Rightarrow v^2 = \frac{G m_1}{R}$$

$$v = \sqrt{\frac{G m_1}{R}}$$

Period and Frequency for Circular Orbits



$$2\pi R = vT \Rightarrow T = \frac{2\pi r}{v} \xrightarrow{v=\sqrt{\frac{Gm_1}{R}}}$$

$$T = \frac{2\pi R}{\sqrt{\frac{Gm_1}{R}}} = \frac{2\pi R\sqrt{R}}{\sqrt{Gm_1}} \Rightarrow$$

$$T^2 = \frac{4\pi^2 R^3}{Gm_1}$$ ← Kepler's 3rd Law for Circular Orbits

$$\frac{T^2}{R^3} = \frac{4\pi^2}{Gm_1}$$ ← Kepler Constant $\approx 3\cdot 10^{-19}\ \frac{s^2}{m^3}$

Mechanical Energy for Circular Orbits

$$\mathcal{E} = K + U \xrightarrow[U=\frac{-Gm_1m_2}{R}]{K=\frac{1}{2}m_2v^2}$$

$$\mathcal{E} = \frac{1}{2}m_2v^2 - \frac{Gm_1m_2}{R} \xrightarrow{v^2=\frac{Gm_1}{R}}$$

$$\mathcal{E} = \frac{1}{2}m_2\frac{Gm_1}{R} - \frac{Gm_1m_2}{R} = \boxed{\frac{-Gm_1m_2}{2R}}$$

Total Energy for an object in a circular orbit

Escape Velocity

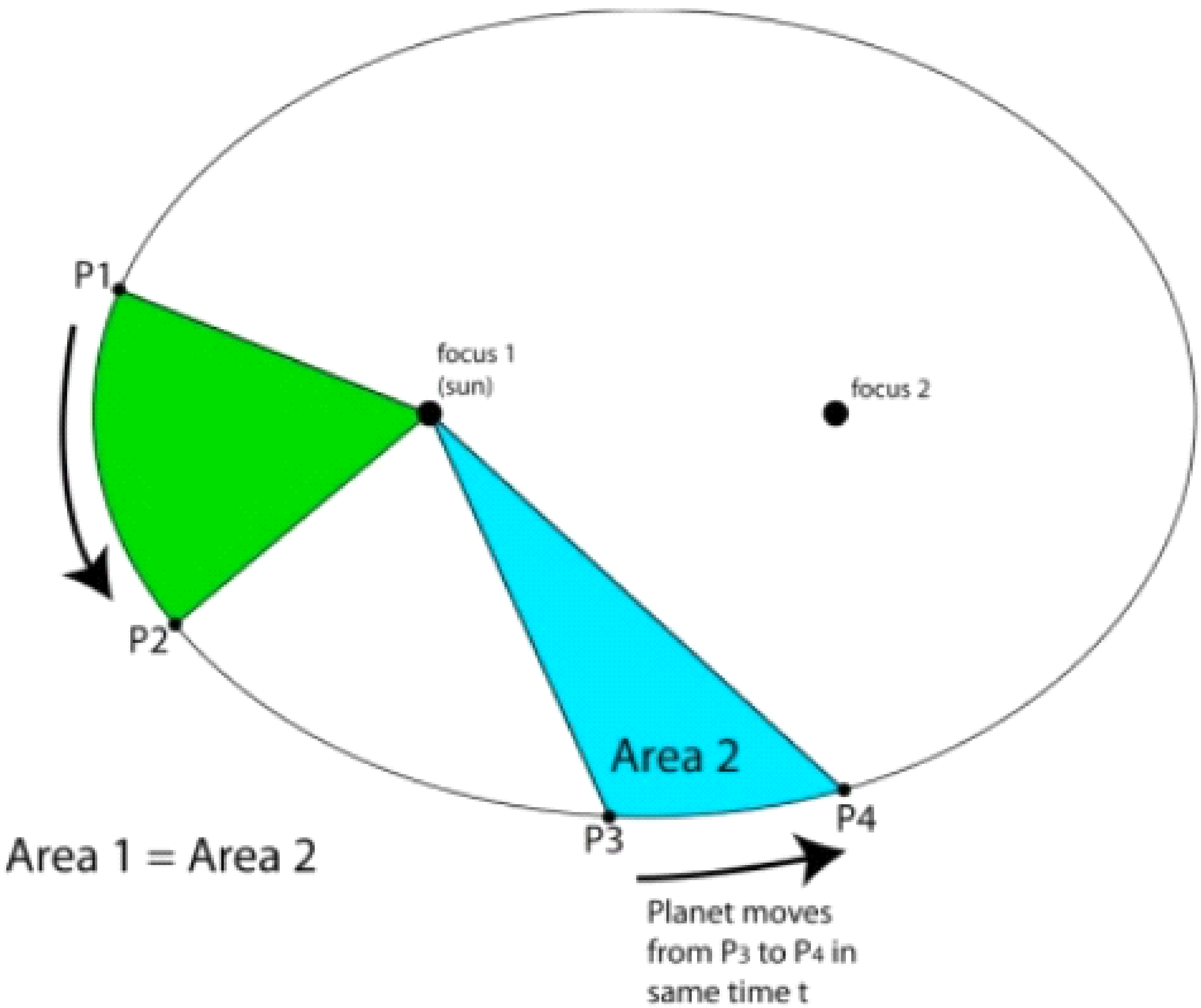
$$\mathcal{E} = K + U = 0 \Rightarrow \frac{1}{2} m_2 v^2 - \frac{G m_1 m_2}{R} = 0$$

$$\Rightarrow \frac{1}{2} m_2 v^2 = \frac{G m_1 m_2}{R} \Rightarrow v^2 = \frac{2 G m_1}{R} \Rightarrow$$

$$\boxed{V_{esc} = \sqrt{\frac{2 G m_1}{R}}}$$

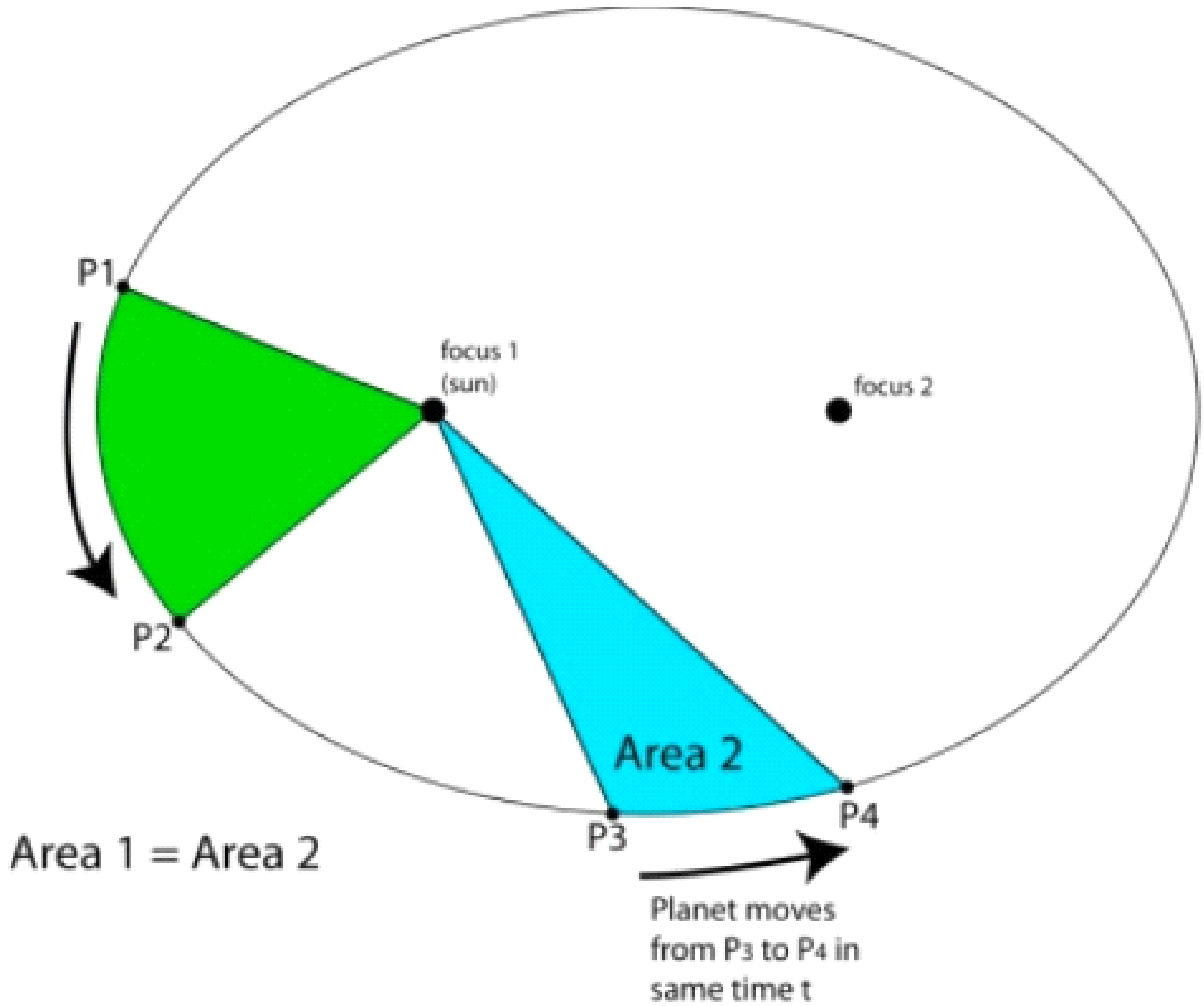
Kepler's Frist Law of Planetary Motion

- The orbits of planetary bodies are ellipses with the sun at one of two foci of the ellipse



Kepler's Second Law of Planetary Motion

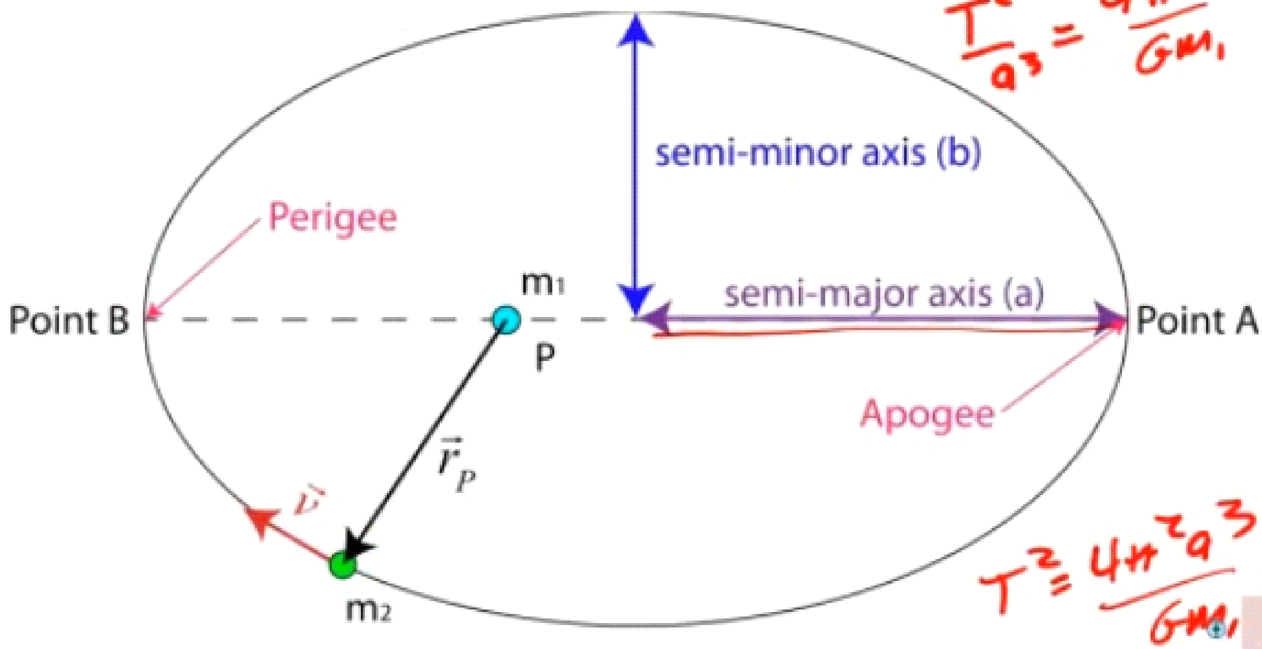
- If you were to draw a line from the sun to the orbiting body, the body would sweep out equal areas along the ellipse in equal amounts of time.



Kepler's Third Law of Planetary Motion

- The ratio of the squares of the periods of two planets is equal to the ratio of the cubes of their semi-major axes.
- The ratio of the squares of the periods to the cubes of their semi-major axes is referred to as Kepler's Constant

$$\frac{T^2}{a^3} = \frac{4\pi^2}{Gm_1}$$

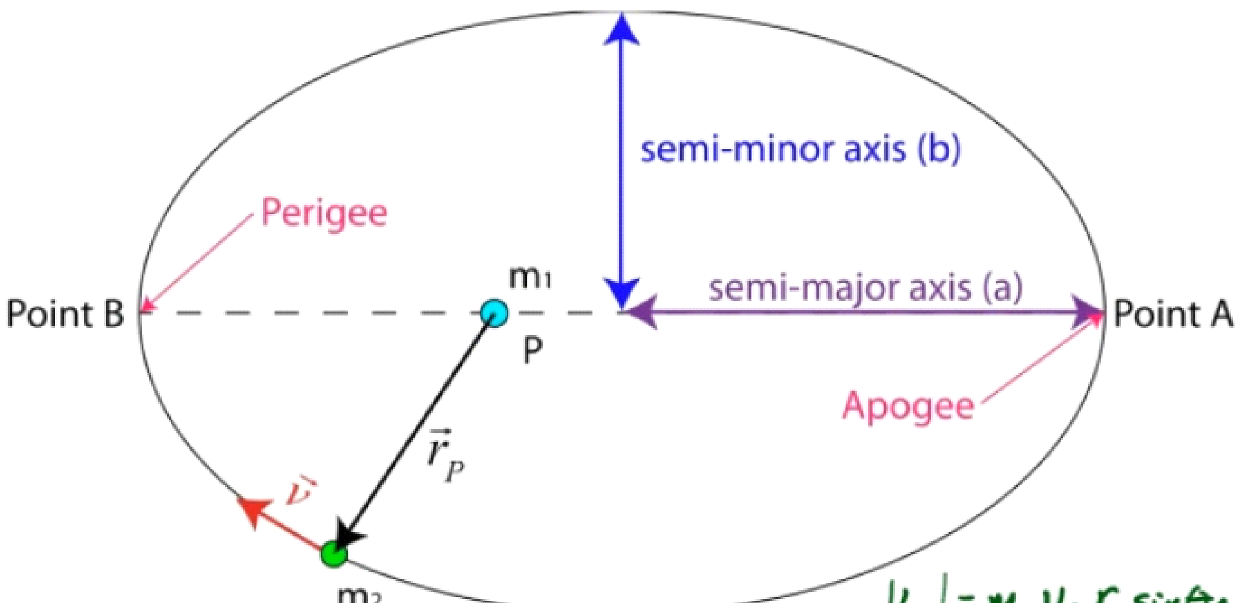


$$T^2 = \frac{4\pi^2 a^3}{Gm_1}$$

Total Mechanical Energy for an Elliptical Orbit

$$E = K + U = \frac{1}{2} m_2 v^2 - \frac{Gm_1 m_2}{r} \Rightarrow E = \frac{-Gm_1 m_2}{2a}$$

Velocity and Radius for an Elliptical Orbit



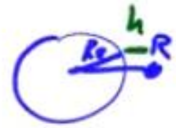
$$|L_p| = m_2 v_A r_a \sin\theta_A = m_2 v_B r_B \sin\theta_B$$

$$\frac{d\vec{L}_p}{dt} = \vec{\tau}_p = 0 \Rightarrow \vec{L}_p \text{ must be conserved}$$

$$@\ A \,\&\, B,\ \theta_A = \theta_B \Rightarrow v_A r_A = v_B r_B$$

Example 1: Rocket Launched Vertically

- A rocket is launched vertically form the surface of the Earth with an initial velocity of 10 km/s. What maximum height does it reach, neglecting air resistance?
- Note that the mass of the Earth (m_1) is 6*10^{24} kg and the radius of the Earth (R_E) is 6.37*10^6 m. You may not assume that the acceleration due to gravity is constant.



$$K_i + U_i = K_f + U_f \Rightarrow \frac{1}{2}m_2 v^2 - \frac{Gm_1m_2}{R_E} = 0 - \frac{Gm_1m_2}{R} \Rightarrow$$

$$v^2 - \frac{2Gm_1}{R_E} = -\frac{2Gm_1}{R} \Rightarrow \frac{1}{R} = \frac{-v^2}{2Gm_1} + \frac{2Gm_1}{2Gm_1R_E} \Rightarrow \frac{1}{R} = \frac{-v^2}{2Gm_1} + \frac{1}{R_E} \Rightarrow$$

$$\frac{1}{R} = \frac{-(10{,}000)^2}{2(6.67\cdot10^{-11})(6\cdot10^{24})} + \frac{1}{6.37\cdot10^6} \Rightarrow R = 3.12\cdot10^7\,m$$

$$h = R - R_E = 3.12\cdot10^7\,m - 6.37\cdot10^6\,m = \boxed{2.48\cdot10^7\,m}$$

2007 Free Response Question 2

Mech. 2.

In March 1999 the Mars Global Surveyor (GS) entered its final orbit about Mars, sending data back to Earth. Assume a circular orbit with a period of 1.18×10^2 minutes = 7.08×10^3 s and orbital speed of 3.40×10^3 m/s. The mass of the GS is 930 kg and the radius of Mars is 3.43×10^6 m.

(a) Calculate the radius of the GS orbit.

(b) Calculate the mass of Mars.

(c) Calculate the total mechanical energy of the GS in this orbit.

(d) If the GS was to be placed in a lower circular orbit (closer to the surface of Mars), would the new orbital period of the GS be greater than or less than the given period?

_______Greater than   _________Less than

Justify your answer.

(e) In fact, the orbit the GS entered was slightly elliptical with its closest approach to Mars at 3.71×10^5 m above the surface and its furthest distance at 4.36×10^5 m above the surface. If the speed of the GS at closest approach is 3.40×10^3 m/s, calculate the speed at the furthest point of the orbit.

a) 

$T = 7080\,s$
$m = 930\,kg$
$v = 3400\,\frac{m}{s}$

$$T = \frac{2\pi r}{v} \Rightarrow r = \frac{vT}{2\pi} = \frac{(3400)(7080)}{2\pi}$$

$$\Rightarrow \boxed{r = 3.83\cdot 10^6\,m}$$

b) $$F_{NETc} = \frac{mv^2}{r} = \frac{Gm_1m_2}{r^2} \Rightarrow v^2 = \frac{Gm_{mars}}{r} \Rightarrow$$

$$m_{mars} = \frac{rv^2}{G} = \frac{(3.83\cdot 10^6\,m)(3400\,\frac{m}{s})^2}{6.67\cdot 10^{-11}\,\frac{N\cdot m^2}{kg^2}} = \boxed{6.64\cdot 10^{23}\,kg}$$

c) $$E_{TOT} = K + U = \frac{1}{2}mv^2 - \frac{Gm_sm_{mars}}{r} = \frac{1}{2}(930)(3400)^2 - \frac{(6.67\cdot 10^{-11})(930)(6.64\cdot 10^{23})}{3.83\cdot 10^6}$$

$$E_{TOT} = \boxed{-5.38\cdot 10^9\,J}$$

d) Less Than $v\uparrow$ $d\downarrow$ $\vec{v} = \frac{d}{t} \Rightarrow t = \frac{d}{v}$ $\begin{matrix} d\downarrow \\ v\uparrow \end{matrix}$

both lead to a shorter T

Sample Questions

Wednesday, April 5, 2017 5:57 PM

Question 6

> If the resultant of all the external forces acting on a system of particles is zero, then from an inertial frame,
> one can surely say that:
> 1: Linear momentum of the system does not change in time.
> 2: Kinetic energy of the system does not change in time.
> 3: Angular momentum of the system does not change in time.
> 4: Potential energy of the system does not change in time.

My attempt (I will for moment assume that my system of particle might be a rod)

I can surely see that since net external force has a resultant 0 it easily implies that Linear momentum remain conserved. (A)

The net external forces may be 0 but the net torque may or may not be 0 hence the body may start to rotate about some axis and gain rotational kinetic energy, hence answer (B) is incorrect.

Also, Following from above net torque acting on system may not be zero Angular momentum is not conserved. (C) is incorrect.

Equilibrium Summary

- There are two necessary conditions for equilibrium
- The resultant external force must equal zero:
  $$\Sigma \mathbf{F} = 0$$
  - This is a statement of translational equilibrium
  - The acceleration of the center of mass of the object must be zero when viewed from an inertial frame of reference
- The resultant external torque about ***any*** axis must be zero: $\Sigma\tau = 0$
  - This is a statement of rotational equilibrium
  - The angular acceleration must equal zero

1998 Multiple Choice

Thursday, March 23, 2017 11:00 PM

Question 17

$$U = -\int_{ref}^{r} \vec{F} \cdot d\vec{r}$$

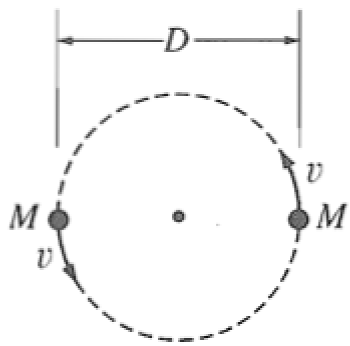
Example

Why negative

$$F(x) = -\frac{dU}{dx}$$

Example

Question 20



20. Two identical stars, a fixed distance D apart, revolve in a circle about their mutual center of mass, as shown above. Each star has mass M and speed v. G is the universal gravitational constant. Which of the following is a correct relationship among these quantities?

(A) $v^2 = GM/D$

(B) $v^2 = GM/2D$

(C) $v^2 = GM/D^2$

(D) $v^2 = MGD$

(E) $v^2 = 2GM^2/D$

$$F_{NET_C} = \frac{mv^2}{r} \Rightarrow \frac{GMM}{D^2} = \frac{Mv^2}{D/2} \Rightarrow v^2 = \frac{GMMD}{D^2\, 2M} \Rightarrow$$

$$v^2 = \frac{GM}{2D} \quad \text{B}$$

Question 33

$$P = \tau\,\omega$$

$$\text{power} = \frac{\text{force} \times \text{linear distance}}{\text{time}}$$

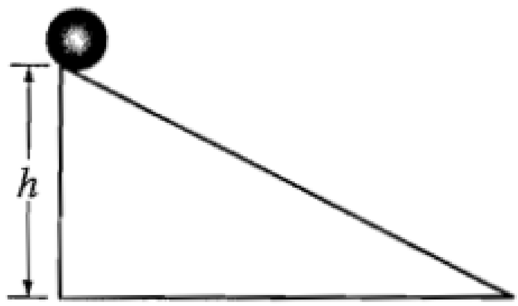
$$= \frac{\left(\frac{\text{torque}}{r}\right) \times (r \times \text{angular speed} \times t)}{t}$$

$$= \text{torque} \times \text{angular speed.}$$

2004 Multiple Choice

Wednesday, April 19, 2017 12:58 AM

Question 8



A sphere of mass M, radius r, and rotational inertia I is released from rest at the top of an inclined plane of height h as shown above.

Conservation of Energy

$K = \frac{1}{2}mv^2$

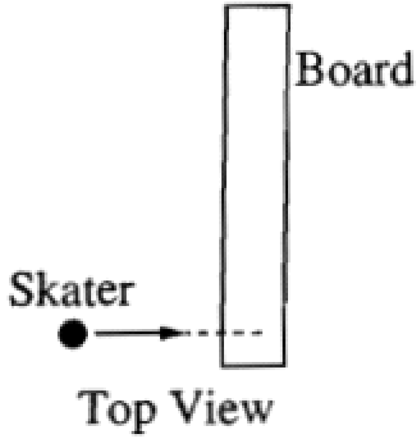
$K_{rot} = \frac{1}{2}I\omega^2$

$v_{cm} = r\omega$

We now can consider the kinetic energy in two parts, rotation and translation of the center of mass. Then $Mgh = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\left(\frac{v_{cm}}{r}\right)^2$

Solving for v_{cm} yields $\sqrt{\dfrac{2Mghr^2}{I + Mr^2}}$

Question 17

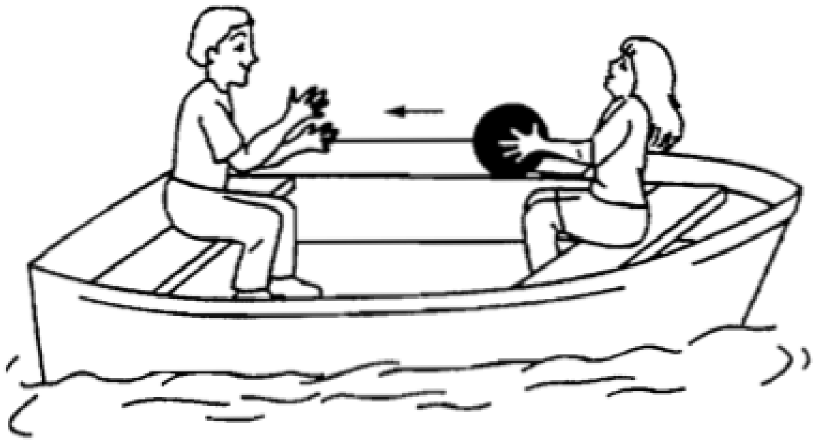


A long board is free to slide on a sheet of frictionless ice. As shown in the top view above, a skater skates to the board and hops onto one end, causing the board to slide and rotate. In this situation, which of the following occurs?

(A) Linear momentum is converted to angular momentum.
(B) Kinetic energy is converted to angular momentum.
(C) Rotational kinetic energy is conserved.
(D) Translational kinetic energy is conserved.
(E) Linear momentum and angular momentum are both conserved.

- No net external torque --> no change in angular momentum
- No net external force --> no change in linear momentum

Question 25



25. As shown above, two students sit at opposite ends of a boat that is initially at rest. The student in the front throws a heavy ball to the student in the back. What is the motion of the boat at the time immediately after the ball is thrown and, later, after the ball is caught? (Assume that air and water friction are negligible.)

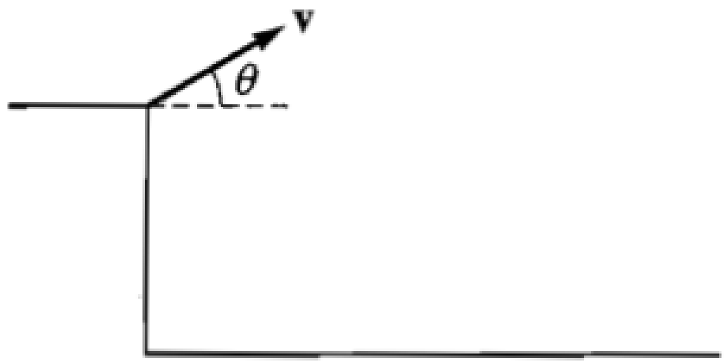
- After ball tossed it has momentum toward back so the boat must have forward momentum to maintain the zero initial momentum.
- After the catch both the ball and boat the two must be at the same speed and this must be at rest to have momentum zero as it was originally

Question 29

$$T = 2\pi\sqrt{\frac{M}{k}}$$

- Neither the mass, M, nor the spring constant, k, depend on the gravitational force of the planet.

Question 33



$$\text{Time of flight, } t = \frac{2v_0 \sin\theta}{g}$$

$$\text{Maximum height reached, } H = \frac{v_0^2 \sin^2\theta}{2g}$$

$$\text{Horizontal range, } R = \frac{v_0^2 \sin 2\theta}{g}$$

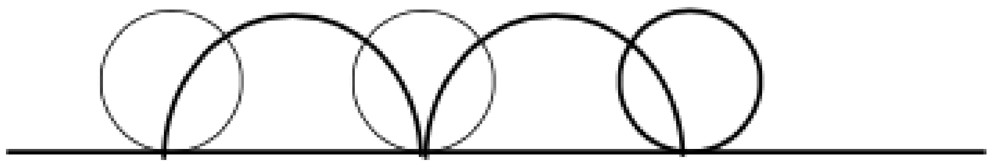
The 45° angle gives the maximum horizontal travel to the original elevation, but the smaller angle causes the projectile to have a greater horizontal component of velocity, so given the additional time of travel allows such a trajectory to advance a greater horizontal distance. In other words given enough time the smaller angle of launch gives a parabola which will eventual cross the parabola of the 45° launch.

Question 34

A car travels forward with constant velocity. It goes over a small stone, which gets stuck in the groove of a tire. The initial acceleration of the stone, as it leaves the surface of the road, is

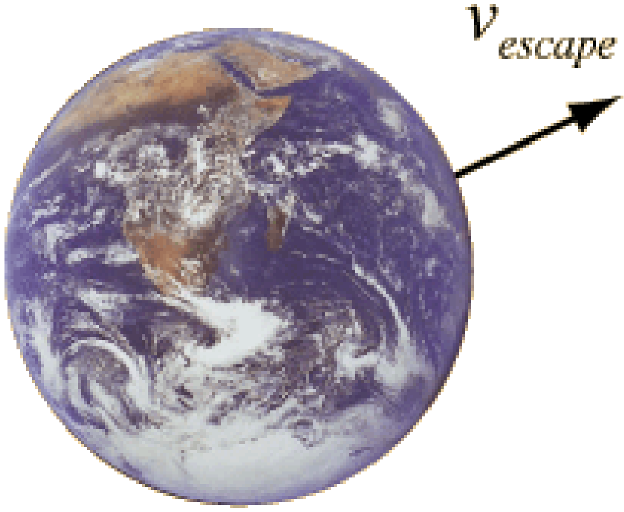
(A) vertically upward
(B) horizontally forward
(C) horizontally backward
(D) zero
(E) upward and forward, at approximately 45° to the horizontal

- The path of a point on the rim of a non-slipping rolling wheel is a cycloid.



Question 35

- Escape speed



$$v_{escape} = 11.2\ km/s$$

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

$$v_{escape} = \sqrt{\frac{2GM}{r}}$$

- Orbital speed

$$\frac{GMm}{R^2} = \frac{mv^2}{R} \Longrightarrow v = \sqrt{\frac{GM}{R}}$$

2009 Multiple Choice

Wednesday, April 19, 2017 1:11 AM

Question 7

$$F_{\text{net}} = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = v\frac{\Delta m}{\Delta t}$$

Question 11

11. A student is asked to determine the mass of Jupiter. Knowing which of the following about Jupiter and one of its moons will allow the determination to be made?

I. The time it takes for Jupiter to orbit the Sun
II. The time it takes for the moon to orbit Jupiter
III. The average distance between the moon and Jupiter

An application of Kepler's Third Law;

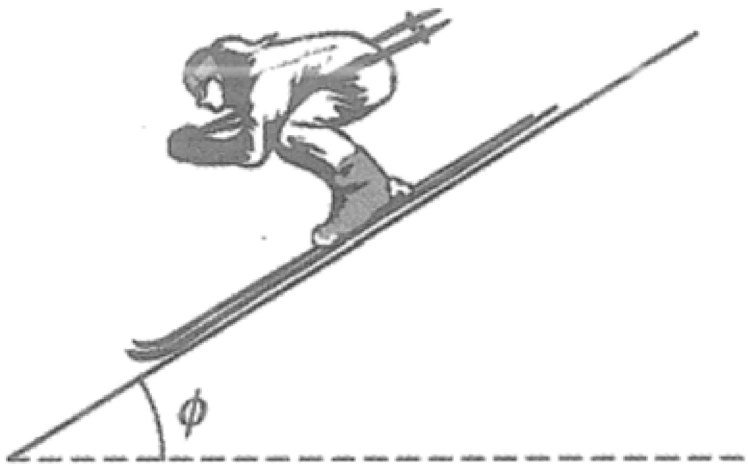
$$\frac{r^3}{T^2} = \frac{GM}{4\pi}$$

If you know both r and T, you can find the mass M. Alternatively, use the satellite equation (for a circular orbit, which may or may not be true):

$$F_{\text{net}} = \frac{mv^2}{r} = \frac{GMm}{r^2} \Rightarrow v^2 = \frac{GM}{r}$$

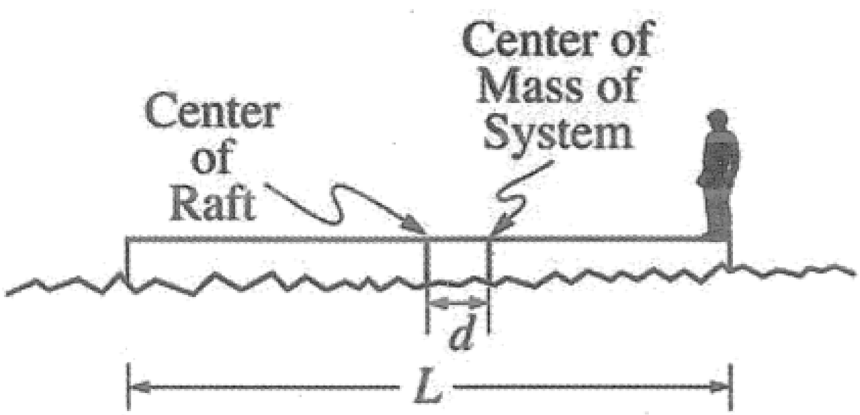
For a circular orbit, $v = 2\pi r/T$, so you recover Kepler's Third Law (which is true even for elliptical orbits, with a, the ellipse's semimajor axis, replacing the radius r.)

Question 19



- When a=0, μ= tanθ

Question 21

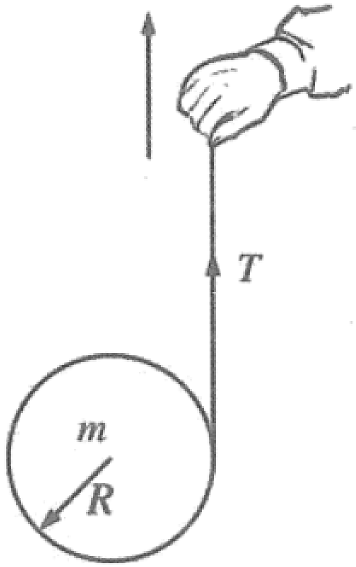


A person is standing at one end of a uniform raft of length L that is floating motionless on water, as shown above. The center of mass of the person-raft system is a distance d from the center of the raft. The person then walks to the other end of the raft. If friction between the raft and the water is negligible, how far does the raft move relative to the water?

- After the person has walked to the other end, the situation must be the mirror image of this one; the center of mass will be shifted to a distance d to the left of the center of the raft.
- But the center of mass cannot move as the person walks on the boat.

- Therefore, the boat must have moved a distance 2d to the right.

Question 24



A solid cylinder of mass m and radius R has a string wound around it. A person holding the string pulls it vertically upward, as shown above, such that the cylinder is suspended in midair for a brief time interval Δt and its center of mass does not move. The tension in the string is T, and the rotational inertia of the cylinder about its axis is $\frac{1}{2}mR^2$.

24. The linear acceleration of the person's hand during the time interval Δt is

(A) $\frac{T - mg}{m}$

(B) $2g$

(C) $\frac{g}{2}$

(D) $\frac{T}{m}$

(E) zero

(E) zero

There is an associated, famous problem about dropping a spool (with the cord held by a *motionless* hand). In that case, you write Newton's Second Law for the forces

$$F_{\text{net}} = mg - T = ma$$

and for the torques,

$$\tau = TR = I\alpha = I(a/R)$$

where one associates a with a_{cm}. Then $T = Ia/R^2$ and

$$ma = mg - Ia/R^2 \quad \Rightarrow \quad a = \frac{mg}{m + (I/R^2)}$$

The problem here is, of course, that the center of mass acceleration is *zero*. In fact you have to think a little more carefully. The typical relationship for rolling without slipping, $a = \alpha R$, still holds, but now it is the *tangential* acceleration of the rim, which is equal to the upward acceleration of the cord (and the hand.) Generally in rolling without slipping (on a horizontal surface, for example), the rim's acceleration is the same as the center of mass's; that isn't true here. Here, the center of mass doesn't move at all during the time interval Δt. But we can still say, with a the acceleration of the rim (equal to the acceleration of the hand),

$$\tau = TR = I\alpha = I(a/R)$$

From the previous problem, we know $T = mg$. Then

$$mgR = I(a/R) = \tfrac{1}{2}mR^2 \times (a/R) = \tfrac{1}{2}maR$$

Cancel mR on both sides, multiply both sides by 2 and obtain $a = 2g$. Unsurprisingly, 90% of those who took the test got this one wrong.

Question 25

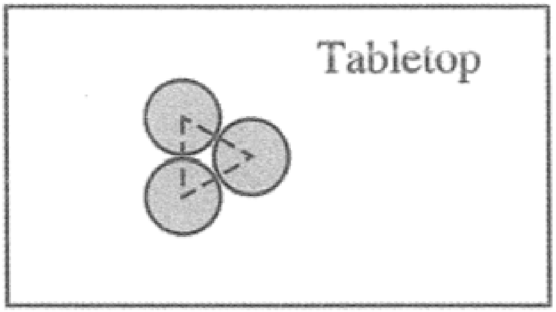
A figure skater goes into a spin with arms fully extended. Which of the following describes the changes in the rotational kinetic energy and angular momentum of the skater as the skater's arms are brought toward the body?

The angular momentum cannot change, because there are no external torques. On the other hand, we have the useful formulas

$$K = \frac{p^2}{2m} \quad \text{and} \quad K = \frac{L^2}{2I}$$

As the skater's arms come in, I, which depends on the square of the average distance of mass from the axis, is getting smaller. Since the kinetic energy depends directly on the square of the angular momentum and inversely on the moment of inertia, and I here decreases, it follows K increases. Also, work has to be done to draw the arms in; this work goes into increased kinetic energy.

Question 32



Three identical disks are initially at rest on a frictionless, horizontal table with their edges touching to form a triangle, as shown in the top view above. An explosion occurs within the triangle, propelling the disks horizontally along the surface. Which of the following diagrams shows a possible position of the disks at a later time? (In these diagrams, the triangle is shown in its original position.)

- The center of mass cannot move

Free Response 1999-2001

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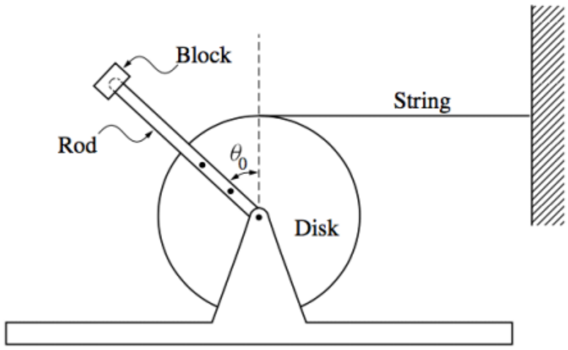
1999 Free Response

Question 3



As shown above, a uniform disk is mounted to an axle and is free to rotate without friction. A thin uniform rod is rigidly attached to the disk so that it will rotate with the disk. A block is attached to the end of the rod. Properties of the disk, rod, and block are as follows.

Disk: mass $= 3m$, radius $= R$, moment of inertia about center $I_D = \frac{3}{2}mR^2$

Rod: mass $= m$, length $= 2R$, moment of inertia about one end $I_R = \frac{4}{3}mR^2$

Block: mass $= 2m$

The system is held in equilibrium with the rod at an angle θ_0 to the vertical, as shown above, by a horizontal string of negligible mass with one end attached to the disk and the other to a wall. Express your answers to the following in terms of m, R, θ_0, and g.

(a) Determine the tension in the string.

For indicating that the net torque is zero, or that the clockwise and counterclockwise torques are equal

For a correct expression for the torque exerted by the rod

$\tau_{rod} = mgR\sin\theta_0$

For a correct expression for the torque exerted by the block

$\tau_{block} = 2mg(2R)\sin\theta_0 = 4mgR\sin\theta_0$

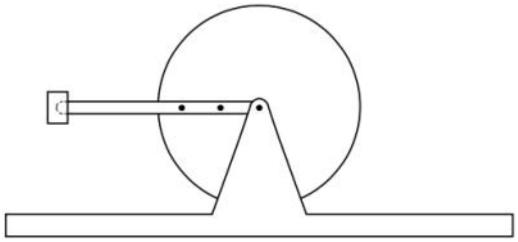
For a correct expression for the torque exerted by the string

$\tau_{string} = TR$

For adding the counterclockwise torques and setting the sum equal to the clockwise torque (this point not awarded for just one torque)

$TR = 4mgR\sin\theta_0 + mgR\sin\theta_0$

$T = 5mg\sin\theta_0$



As the disk rotates, the rod passes the horizontal position shown above.

(c) Determine the linear speed of the mass at the end of the rod for the instant the rod is in the horizontal position.

For indicating that energy is conserved

For indicating that the potential energy of two bodies (the rod and the block) changes

$\Delta U = mgh_{rod} + mgh_{block}$

For the correct expressions for these two potential energies

$\Delta U = mgR\cos\theta_0 + 2mg(2R)\cos\theta_0$

For indicating the correct kinetic energy when the rod is horizontal

$\boxed{K = \frac{1}{2}I\omega^2}$

Equating the kinetic and potential energies, and solving for the angular speed

$$\frac{1}{2}\left(\frac{65}{6}mR^2\right)\omega^2 = mgR\cos\theta_0 + 4mgR\cos\theta_0$$

$$\omega = \sqrt{\frac{12g\cos\theta_0}{13R}}$$

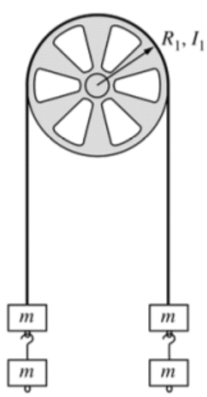
For using the relationship between linear and angular speed, and substituting ω and the correct radius, $2R$

$v = \omega r$

$$v = \left(\sqrt{\frac{12g\cos\theta_0}{13R}}\right)(2R) = 4\sqrt{\frac{3gR\cos\theta_0}{13}}$$

2000 Free Response

Question 2



Mech 3.

A pulley of radius R_1 and rotational inertia I_1 is mounted on an axle with negligible friction. A light cord passing over the pulley has two blocks of mass m attached to either end, as shown above. Assume that the cord does not slip on the pulley. Determine the answers to parts (a) and (b) in terms of m, R_1, I_1, and fundamental constants.

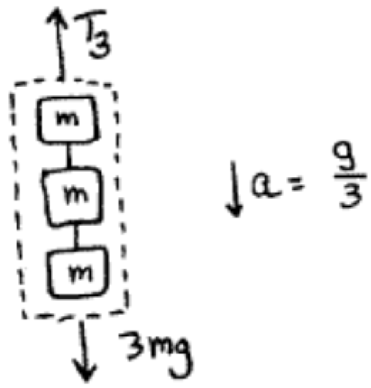
(b) One block is now removed from the right and hung on the left. When the system is released from rest, the three blocks on the left accelerate downward with an acceleration $\frac{g}{3}$. Determine the following.

i. The tension T_3 in the section of cord supporting the three blocks on the left

ii. The tension T_1 in the section of cord supporting the single block on the right

iii. The rotational inertia I_1 of the pulley

i. 2 points



$\Sigma F = ma$

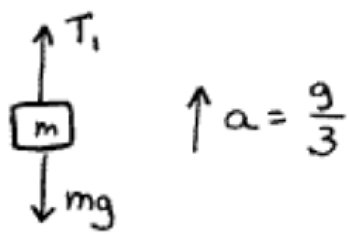
For correct substitutions into Newton's second law

$$3mg - T_3 = 3m\left(\frac{g}{3}\right)$$

For a correct solution for T_3

$$T_3 = 2mg$$

ii. 2 points



$\Sigma F = ma$

For correct substitutions into Newton's second law

$$T_1 - mg = m\left(\frac{g}{3}\right)$$

For a correct solution for T_1

$$T_1 = \frac{4}{3}mg$$

iii. 4 points

For $\tau = I_1\alpha$

For $\alpha = \frac{a}{R_1} = \frac{g}{3R_1}$

For $\tau = (T_3 - T_1)R_1$

For correct substitutions into $\tau = I_1\alpha$ and solution for I_1

$$\left(2mg - \frac{4}{3}mg\right)R_1 = I_1\left(\frac{g}{3R_1}\right)$$

$$I_1 = 2mR_1^{\,2}$$

Alternate Solution

Use conservation of energy, $\Delta E = \Delta K + \Delta U = 0$

For $\Delta K = -\Delta U$

For $\Delta K = \frac{1}{2}mv^2 + \frac{1}{2}(3m)v^2 + \frac{1}{2}I_1\omega^2$, where $\omega = \frac{v}{R_1}$

For $\Delta U = mgh - 3mgh = -2mgh$, where $h = \frac{v^2}{2a} = \frac{3v^2}{2g}$

For correct substitutions and solution for I_1

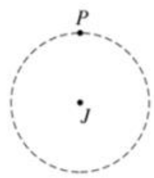
$$I_1 = 2mR_1^{\,2}$$

2001 Free Response

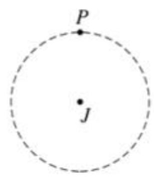
Question 2

(c) Suppose that the injection of the satellite into orbit is less than perfect. For an injection velocity that differs from the desired value in each of the following ways, sketch the resulting orbit on the figure. (J is the center of Jupiter, the dashed circle is the desired orbit, and P is the injection point.) Also, describe the resulting orbit qualitatively but specifically.

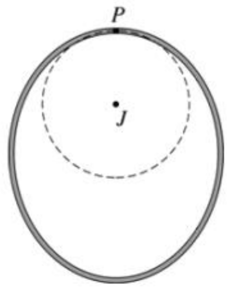
i. When the satellite is at the desired altitude over the equator, its velocity vector has the correct direction, but the speed is slightly faster than the correct speed for a circular orbit of that radius.



ii. When the satellite is at the desired altitude over the equator, its velocity vector has the correct direction, but the speed is slightly slower than the correct speed for a circular orbit of that radius.



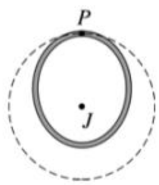
2. (c) i. **3 points**



For stating that the orbit is an ellipse	**1 point**
For diagram with orbit drawn completely outside the circle with point of contact only at point P and major axis along PJ.	**2 points**

Partial credit of **1 point** awarded for any path or orbit completely outside the circle.
No points were awarded in any part of path or orbit was inside the circle.

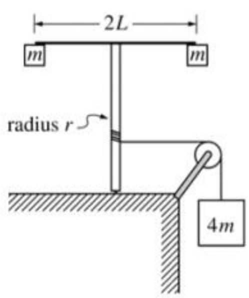
2. (c) ii. **3 points**



	Distribution of Points
For stating that the orbit is an ellipse	**1 point**
For diagram with orbit drawn completely inside the circle with point of contact only at point P and major axis along PJ.	**2 points**

Partial credit of **1 point** awarded for any path or orbit completely inside the circle.
No points were awarded if any part of path or orbit was outside the circle.

Question 3



Experiment A

Mech 3.

A light string that is attached to a large block of mass $4m$ passes over a pulley with negligible rotational inertia and is wrapped around a vertical pole of radius r, as shown in Experiment A above. The system is released from rest, and as the block descends the string unwinds and the vertical pole with its attached apparatus rotates. The apparatus consists of a horizontal rod of length $2L$, with a small block of mass m attached at each end. The rotational inertia of the pole and the rod are negligible.

(b) Determine the downward acceleration of the large block.

3. (b) **6 points**

For a correct expression of Newton's 2nd law

$F = ma$

For correct substitutions into Newton's law

$4mg - T = 4ma$

For a correct formula for torque

$\tau = I\alpha$ or Tr

$I\alpha = Tr$

$T = \frac{I\alpha}{r}$

From Newton's 2nd law equation above:

$T = 4mg - 4ma$

Substituting into the torque equation:

$\frac{I\alpha}{r} = 4mg - 4ma$

For substituting the expression for I from part (a) into Newton's law

$\frac{2mL^2\alpha}{r} = 4mg - 4ma$

For the expression $\alpha = a/r$

Substituting this expression into the previous equation:

$\frac{2mL^2 a}{r^2} = 4mg - 4ma$

For the correct answer

$a = \frac{2gr^2}{L^2 + 2r^2}$

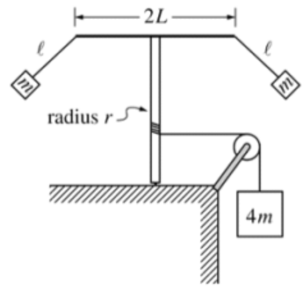
(c) When the large block has descended a distance D, how does the instantaneous total kinetic energy of the three blocks compare with the value $4mgD$? Check the appropriate space below.

____ Greater than $4mgD$ ____ Equal to $4mgD$ ____ Less than $4mgD$

3. (c) **3 points**

For correctly checking the space in front of "Equal to $4mgD$"

For correct justification, such as "The kinetic energy gained by the two smaller blocks comes from the decrease in the potential energy of the $4m$ block." OR "Total energy is conserved."



Experiment B

The system is now reset. The string is rewound around the pole to bring the large block back to its original location. The small blocks are detached from the rod and then suspended from each end of the rod, using strings of length ℓ. The system is again released from rest so that as the large block descends and the apparatus rotates, the small blocks swing outward, as shown in Experiment B above. This time the downward acceleration of the block decreases with time after the system is released.

(d) When the large block has descended a distance D, how does the instantaneous total kinetic energy of the three blocks compare to that in part (c) ? Check the appropriate space below.

____ Greater ____ Equal ____ Less

3. (d) **3 points**

For correctly checking the space in front of "Less"
For correct justification, such as "The small blocks rise and gain potential energy. The total energy available is still $4mgD$. Therefore the kinetic energy must be less than in part (c)."

Free Response 2002-2004

2017年5月8日 星期一 上午1:38

2002 Free Response

Question 2

Mech 2.

The cart shown above is made of a block of mass m and four solid rubber tires each of mass $m/4$ and radius r. Each tire may be considered to be a disk. (A disk has rotational inertia $\frac{1}{2}ML^2$, where M is the mass and L is the radius of the disk.) The cart is released from rest and rolls without slipping from the top of an inclined plane of height h. Express all algebraic answers in terms of the given quantities and fundamental constants.

(b) Determine the speed of the cart when it reaches the bottom of the incline.

(b) 7 points

For an indication of the conservation of mechanical energy

$E_{top} = E_{bottom}$; $\Delta U = -\Delta K$; or equivalent

For correct expressions for energies at the top

$$K_{top} = 0;\ U_{top} = mgh + 4\left(\frac{1}{4}mgh\right) = 2mgh$$

For a correct expression for potential energy at the bottom and for recognizing that kinetic energy at the bottom is the sum of translational and rotational kinetic energies

$$U_{bottom} = 0;\ \boxed{K_{bottom} = K_{trans} + K_{rot}}$$

For a correct expression for translational kinetic energy at the bottom

$$K_{trans} = \frac{1}{2}(2m)v^2 = mv^2$$

For a correct expression for rotational kinetic energy at the bottom

$$K_{rot} = \frac{1}{2}I\omega^2$$

For recognition of the relationship between translational and rotational velocity

$$\omega = \frac{v}{r}$$

Substituting these expressions to determine total kinetic energy at the bottom

$$\boxed{K_{bottom} = mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\frac{v^2}{r^2} = \frac{5}{4}mv^2}$$

Setting potential energy at the top equal to the kinetic energy at the bottom

$$\frac{5}{4}mv^2 = 2mgh$$

For the correct solution for v

$$v = \sqrt{\frac{8}{5}gh}$$

Question 3

(c) Suppose that the object is released from rest at the origin. Determine the speed of the particle at $x = 2$ m.

In the laboratory, you are given a glider of mass 0.5 kg on an air track. The glider is acted on by the force determined in part (b). Your goal is to determine experimentally the validity of your theoretical calculation in part (c).

(d) From the list below, select the additional equipment you will need from the laboratory to do your experiment by checking the line next to each item. If you need more than one of an item, place the number you need on the line.

___ Meterstick ___ Stopwatch ___ Photogate timer ___ String ___ Spring

___ Balance ___ Wood block ___ Set of objects of different masses

(e) Briefly outline the procedure you will use, being explicit about what measurements you need to make in order to determine the speed. You may include a labeled diagram of your setup if it will clarify your procedure.

(d) 2 points

For indicating items of equipment consistent with the procedure described in part (e) (at least two of the items if more than two were used)

Note: If part (e) was not attempted, only 1 point maximum was awarded. Unreasonable indications, such as all the items being checked, were not awarded any points.

(e) 3 points

For a complete description of any correct procedure.

Partial credit was awarded for less complete descriptions. The following were common examples. Other examples, though rarely cited, could receive partial or full credit.

1. Using photogates

Place the photogates near $x = 2$ m and a small distance apart (such as a glider length). Measure the distance between the photogates. Measure the time the glider takes to travel between the photogates. Obtain the speed from distance/time.

Note: No points were given if the distance measured was from 0 to 2 m and the time to travel 2 m was used.

2. Using a spring

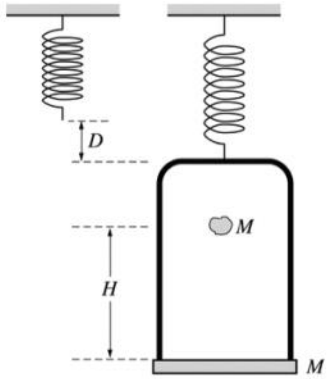
The spring constant k of the spring must be known, or if not, then measured. Set up the spring at $x = 2$ m so that it is compressed when struck by the glider. Measure the distance of maximum compression x_m. The velocity can then be determined from the equation $\frac{1}{2}kx_m = \frac{1}{2}mv^2$.

3. Treating the glider as a projectile

Adjust the starting point so that $x = 2$ m is at end of the track. Thus the glider leaves the track at this point and becomes a projectile. The height of the track determines the time interval t that the glider is in the air. The horizontal distance x from the end of the track to the point where the glider hits the ground is measured and then the velocity is computed from x/t.

2003 Free Response

Question 2



Mech. 2.

An ideal spring is hung from the ceiling and a pan of mass M is suspended from the end of the spring, stretching it a distance D as shown above. A piece of clay, also of mass M, is then dropped from a height H onto the pan and sticks to it. Express all algebraic answers in terms of the given quantities and fundamental constants.

(c) Determine the period of the simple harmonic motion that ensues.

For use of the correct equation for the period of a mass on a spring

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

For recognition that $m = 2M$

For correct calculation of k using the force equation for the initial stretching of the spring

$$\boxed{Mg = kD\text{, giving } k = \frac{Mg}{D}}$$

For the correct answer after substituting for m and k

$$T = 2\pi\sqrt{\frac{2D}{g}}$$

Question 3

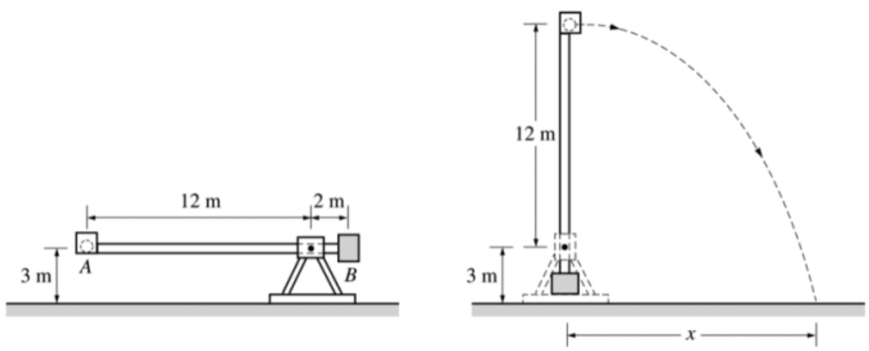


Figure 1 Figure 2

Mech. 3.

Some physics students build a catapult, as shown above. The supporting platform is fixed firmly to the ground. The projectile, of mass 10 kg, is placed in cup A at one end of the rotating arm. A counterweight bucket B that is to be loaded with various masses greater than 10 kg is located at the other end of the arm. The arm is released from the horizontal position, shown in Figure 1, and begins rotating. There is a mechanism (not shown) that stops the arm in the vertical position, allowing the projectile to be launched with a horizontal velocity as shown in Figure 2.

(b) The students assume that the mass of the rotating arm, the cup, and the counterweight bucket can be neglected. With this assumption, they develop a theoretical model for x as a function of the counterweight mass using the relationship $x = v_x t$, where v_x is the horizontal velocity of the projectile as it leaves the cup and t is the time after launch.

i. How many seconds after leaving the cup will the projectile strike the ground?

ii. Derive the equation that describes the gravitational potential energy of the system relative to the ground when in the position shown in Figure 1, assuming the mass in the counterweight bucket is M.

iii. Derive the equation for the velocity of the projectile as it leaves the cup, as shown in Figure 2.

ii. (3 points)

For determining the potential energy of both the load in the counterweight bucket and the projectile

For the correct value of the potential energy of the bucket load

$U_b = mgh = M(9.8)3 = 29.4M$

For the correct value of the potential energy of the projectile

$U_p = mgh = (10)(9.8)3 = 294$

$U_{init} = U_b + U_p = 29.4M + 294$ (or $30M + 300$ using $g = 10\ \mathrm{m/s^2}$)

iii. (5 points)

For a valid statement or equation indicating conservation of energy	1 point
$\boxed{U_{init} = U_{final} + K}$	
For the correct final potential energy of the bucket load	1 point
For the correct final potential energy of the projectile	1 point
$U_{final} = M(9.8)(1) + (10)(9.8)(15) = 9.8M + 1470$	
For having terms for the final kinetic energy of both the bucket load and the projectile	1 point
$K_p = (1/2)10\upsilon_x^2$ and $K_b = (1/2)M\upsilon_b^2$ OR $K_p = (1/2)(1440)\omega^2$ and $K_b = (1/2)(4M)\omega^2$	
For using one of the following relationships to write all expressions in terms of υ_x	1 point
$\upsilon_b = (1/6)\upsilon_x$ OR $\boxed{\omega = \upsilon_x/12}$	

Substituting into the conservation of energy equation above and solving for υ_x :

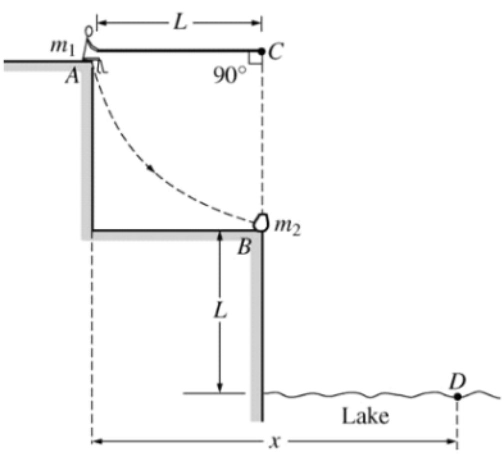
$29.4M + 294 = 9.8M + 1470 + 5\upsilon_x^2 + (M/72)\upsilon_x^2$

$\upsilon_x = \sqrt{(19.6M - 1176)/(5 + (M/72))}$

(or $\sqrt{(20M - 1200)/(5 + (M/72))}$ using $g = 10\ \mathrm{m/s^2}$)

2004 Free Response

Question 1



Mech. 1.

A rope of length L is attached to a support at point C. A person of mass m_1 sits on a ledge at position A holding the other end of the rope so that it is horizontal and taut, as shown above. The person then drops off the ledge and swings down on the rope toward position B on a lower ledge where an object of mass m_2 is at rest. At position B the person grabs hold of the object and simultaneously lets go of the rope. The person and object then land together in the lake at point D, which is a vertical distance L below position B. Air resistance and the mass of the rope are negligible. Derive expressions for each of the following in terms of m_1, m_2, L, and g.

(b) The tension in the rope just before the collision with the object

(b) 4 points

For any indication that there are two forces acting on the person

For an indication that the acceleration of the person is centripetal, i.e. equal to v^2/r or v^2/L

For a correct application of Newton's second law that includes the two forces (tension T and weight) and a non-zero acceleration

$$\boxed{T - m_1 g = \frac{m_1 v_B^2}{r}}$$

$$T = \frac{m_1 v_B^2}{r} + m_1 g$$

For substitution of the expression for v_B from part (a) and L for the radius

$$T = \frac{m_1 (2gL)}{L} + m_1 g = 2m_1 g + m_1 g$$

$$T = 3m_1 g$$

(d) The ratio of the kinetic energy of the person-object system before the collision to the kinetic energy after the collision

(d) 2 points

For correct expressions for the kinetic energy before and after the collision, using the answers to parts (a) and (c)

$$\boxed{\begin{aligned} K_{\text{before}} &= \frac{1}{2} m_1 v_B^2 = \frac{1}{2} m_1 (2gL) = m_1 gL \\ K_{\text{after}} &= \frac{1}{2}(m_1 + m_2) v_{\text{after}}^2 = \frac{1}{2}(m_1 + m_2)\frac{m_1^2}{(m_1 + m_2)^2} 2gL = \frac{m_1^2}{(m_1 + m_2)} gL \end{aligned}}$$

For constructing the ratio $K_{\text{before}}/K_{\text{after}}$ from valid expressions for kinetic energy, in terms of the required quantities. The ratio does not need to be simplified, but if it is the algebra needs to be correct.

$$\frac{K_b}{K_a} = \frac{m_1 gL}{\left(\dfrac{m_1^2 gL}{(m_1 + m_2)}\right)}$$

$$\frac{K_b}{K_a} = \frac{(m_1 + m_2)}{m_1}$$

(e) The total horizontal displacement x of the person from position A until the person and object land in the water at point D.

For a correct expression relating the distance fallen, L, to the time it takes to fall from point B to the water:

$$\boxed{L = \frac{1}{2}gt^2}$$

For indicating that the horizontal displacement from B to D is the answer to part (c) multiplied by the time

$$x_{BD} = v_{\text{after}}t$$

For correctly solving the first equation for t and substituting two quantities into the second equation (this must yield an expression in terms of the required given quantities)

$$\boxed{t = \sqrt{2L/g}}$$

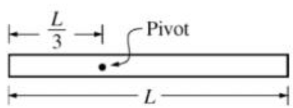
$$x_{BD} = v_{\text{after}}t = \left(\frac{m_1}{m_1 + m_2}\sqrt{2gL}\right)\sqrt{2L/g} = \frac{2m_1L}{m_1 + m_2}$$

For indicating that the total horizontal displacement from A to D is x_{BD} $\boxed{\text{plus } L}$

$$x_{\text{tot}} = x_{BD} + L = \frac{2m_1L}{m_1 + m_2} + L$$

$$x_{\text{tot}} = \frac{(3m_1 + m_2)L}{m_1 + m_2}$$

Question 3

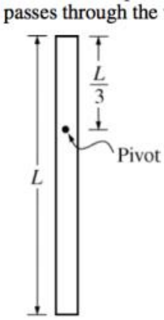


Mech. 3.

A uniform rod of mass M and length L is attached to a pivot of negligible friction as shown above. The pivot is located at a distance $\frac{L}{3}$ from the left end of the rod. Express all answers in terms of the given quantities and fundamental constants.

(a) Calculate the rotational inertia of the rod about the pivot.

(b) The rod is then released from rest from the horizontal position shown above. Calculate the linear speed of the bottom end of the rod when the rod passes through the vertical.



(c) The rod is brought to rest in the vertical position shown above and hangs freely. It is then displaced slightly from this position. Calculate the period of oscillation as it swings.

Alternate Solution

For any statement of the parallel axis theorem

$\boxed{I = I_{cm} + mr^2}$, where r is the distance from the center of mass to the pivot point

For a correct value of the center of mass inertia (calculated or remembered)

$$I_{cm} = \frac{1}{12}ML^2$$

For indicating that $r = L/6$

$$I = \frac{1}{12}ML^2 + M\left(\frac{L}{6}\right)^2$$

For the correct answer

$$I = \frac{ML^2}{9}$$

(b) 7 points

For any indication of conservation of energy

For correctly calculating the change in potential energy of the rod (or the work done on it)

For example: $\Delta U = Mgh_{cm} = Mg\left(\frac{L}{6}\right)$

For writing a conservation equation that includes a rotational kinetic energy (regardless of whether the potential energy is correct)

$$\boxed{\frac{1}{2}I\omega^2 = \frac{MgL}{6}}$$

For any indication that ω is linear speed divided by a distance (regardless of whether the correct distance is used)

Substituting and solving for v:

$$\frac{1}{2}\left(\frac{ML^2}{9}\right)\left(\boxed{\frac{v}{r}}\right)^2 = \frac{MgL}{6}$$

$$\frac{v^2}{r^2} = \frac{MgL}{6}\,\frac{18}{ML^2} = \frac{3g}{L}$$

$$v^2 = \frac{3g}{L}r^2 = \frac{3g}{L}\left(\frac{2L}{3}\right)^2 = \frac{4}{3}gL$$

For the correct answer

$$v = 2\sqrt{\frac{gL}{3}}$$

(c) 4 points

For an equation for the period of a physical pendulum

$$\boxed{T = 2\pi\sqrt{\frac{I}{mgd}}}$$

For substitution of the inertia from part (a)

For indicating that the distance d is the distance from the pivot to the center of mass, i.e. $d = \frac{L}{2} - \frac{L}{3} = \frac{L}{6}$

$$T = 2\pi\sqrt{\frac{ML^2/9}{MgL/6}}$$

For the correct answer

$$T = 2\pi\sqrt{\frac{2L}{3g}}$$

(c) (continued)

(c) (continued)

Alternate solution

For an equation relating the angular acceleration to the torque and inertia

$$\boxed{\alpha = \frac{d^2\theta}{dt^2} = \frac{\tau}{I}}$$

For substituting the inertia from part (a) and the torque as a function of θ

$$\frac{d^2\theta}{dt^2} = \frac{-Mg(L/6)\sin\theta}{ML^2/9} = -\frac{3}{2}\frac{g}{L}\sin\theta$$

For using the approximation $\sin\theta \approx \theta$

$$\frac{d^2\theta}{dt^2} = -\frac{3}{2}\frac{g}{L}\theta$$

Taking $\theta = k\sin\omega t$, the second derivative is $\dfrac{d^2\theta}{dt^2} = -\omega^2 k\sin\omega t$

Substituting into the differential equation and solving for ω:

$$-\omega^2 k\sin\omega t = -\frac{3}{2}\frac{g}{L}k\sin\omega t$$

$$\omega = \sqrt{\frac{3}{2}\frac{g}{L}}$$

Using the relationship between T and ω:

$$T = 2\pi/\omega$$

For the correct answer

$$T = 2\pi\sqrt{\frac{2L}{3g}}$$

Free Response 2005-2007

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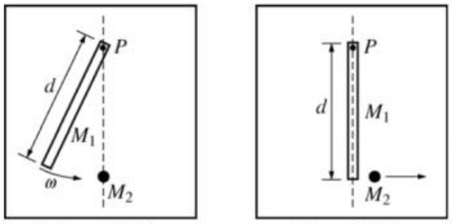
2005 Free Response

Question 3



Before Collision After Collision

TOP VIEWS

Mech. 3.

A system consists of a ball of mass M_2 and a uniform rod of mass M_1 and length d. The rod is attached to a horizontal frictionless table by a pivot at point P and initially rotates at an angular speed ω, as shown above left. The rotational inertia of the rod about point P is $\frac{1}{3}M_1d^2$. The rod strikes the ball, which is initially at rest. As a result of this collision, the rod is stopped and the ball moves in the direction shown above right. Express all answers in terms of M_1, M_2, ω, d, and fundamental constants.

(b) Derive an expression for the speed v of the ball after the collision.

(b) 4 points

For any indication of conservation of angular momentum

$L_b = L_r$

For the correct expression for L_b

For substitution for L_r consistent with part (a)

$$M_2vd = \frac{1}{3}M_1d^2\omega$$

For the correct final expression for v

$$v = \frac{1}{3}\frac{M_1}{M_2}d\omega$$

(c) Assuming that this collision is elastic, calculate the numerical value of the ratio M_1/M_2.

(c) 4 points

For any indication of conservation of kinetic energy

No points were awarded for conservation of mechanical energy.

$\boxed{K_b = K_r}$

For the correct expressions for both kinetic energies

$$\frac{1}{2}M_2 v^2 = \frac{1}{2}I\omega^2$$

$$M_2 v^2 = I\omega^2$$

For correct substitutions for I and for v consistent with part (b)

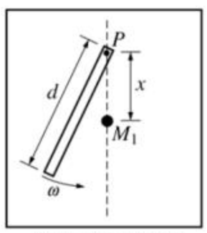
$$M_2\left(\frac{1}{3}\frac{M_1}{M_2}d\omega\right)^2 = \left(\frac{1}{3}M_1 d^2\right)\omega^2$$

$$M_2\frac{1}{9}\left(\frac{M_1}{M_2}\right)^2 d^2\omega^2 = \frac{1}{3}M_1 d^2\omega^2$$

$$\frac{1}{9}\frac{M_1^{\,2}}{M_2} = \frac{1}{3}M_1$$

For the correct final expression for the ratio

$M_1/M_2 = 3$ (3:1 was also acceptable)



Before Collision

(d) A new ball with the same mass M_1 as the rod is now placed a distance x from the pivot, as shown above. Again assuming the collision is elastic, for what value of x will the rod stop moving after hitting the ball?

For the correct equation for conservation of angular momentum

$$M_1 vx = \frac{1}{3}M_1 d^2\omega$$

For solving this equation for v

$$v = \frac{1}{3}\frac{d^2}{x}\omega$$

For the correct equation for conservation of kinetic energy

$$\frac{1}{2}M_1 v^2 = \frac{1}{2}I\omega^2$$

$$M_1 v^2 = \left(\frac{1}{3}M_1 d^2\right)\omega^2$$

$$v^2 = \frac{1}{3}d^2\omega^2$$

For the correct substitution of the above expression for v from momentum conservation into the equation for conservation of kinetic energy

$$\left(\frac{1}{3}\frac{d^2}{x}\omega\right)^2 = \frac{1}{3}d^2\omega^2$$

$$\frac{1}{9}\frac{d^4}{x^2}\omega^2 = \frac{1}{3}d^2\omega^2$$

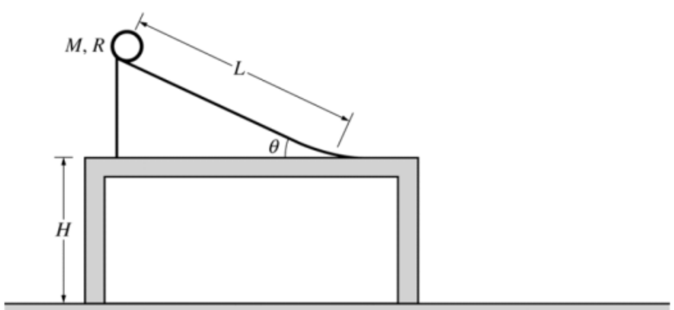
$$\frac{1}{x^2} = \frac{9}{3}\frac{1}{d^2}$$

For the correct final answer

$$x = \frac{d}{\sqrt{3}}$$

2006 Free Response

Question 3



Mech 3.

A thin hoop of mass M, radius R, and rotational inertia MR^2 is released from rest from the top of the ramp of length L above. The ramp makes an angle θ with respect to a horizontal tabletop to which the ramp is fixed. The table is a height H above the floor. Assume that the hoop rolls without slipping down the ramp and across the table. Express all algebraic answers in terms of given quantities and fundamental constants.

(a) Derive an expression for the acceleration of the center of mass of the hoop as it rolls down the ramp.

(a) 5 points

For use of Newton's 2[nd] law in rotational form and the parallel axis theorem

$\boxed{\sum \tau = I\alpha_{cm}}$ and $I = I_{cm} + Mh^2$

For a correct rotational inertia about the point of contact using the parallel axis theorem

$I = MR^2 + MR^2 = 2MR^2$

For a correct torque about the point of contact

$\boxed{\sum \tau = RMg\sin\theta}$

For a correct relationship between linear and angular acceleration for rolling without slipping

$\boxed{\alpha_{cm} = \frac{a_{cm}}{R}}$

Substituting for $\sum \tau$, I, and α_{cm} into the rotational equation above

$RMg\sin\theta = 2MR^2 \frac{a_{cm}}{R}$

For the correct answer

$a_{cm} = \frac{g}{2}\sin\theta$

(d) Suppose that the hoop is now replaced by a disk having the same mass M and radius R. How will the distance from the edge of the table to where the disk lands on the floor compare with the distance determined in part (c) for the hoop?

____ Less than ____ The same as ____ Greater than

Briefly justify your response.

(d) 3 points

For checking the space next to "Greater than"

For a sufficiently detailed justification containing no incorrect statements. Such an answer logically concludes, at a minimum, that the linear speed or velocity at the bottom of the ramp is greater for the disk because the rotational inertia of the disk is less. It is not necessary to state that the time of fall is the same.

One point was awarded for a minimal or partially correct answer.

No justification points were awarded if the space next to "Greater than" was not checked.

Examples of 2-point answers:

A disk will have smaller rotational inertia and will therefore have a greater rotational velocity. This will lead to a greater translational velocity, and a greater distance x.

The rotational inertia is less than the hoop, causing greater acceleration and more final speed at the end of the table.

The acceleration when $I = MR^2/2$ is $(2/3)g\sin\theta$, so the disk will be moving faster at the bottom of the ramp and will travel farther.

Examples of 1-point answers:

A disk has a larger rotational inertia, so it will have a greater kinetic energy and will therefore land farther from the ramp.

The moment of inertia for the disk is smaller, thus its rotational velocity is bigger, causing it to go further.

Less energy will be used to spin the disk than the hoop, and I of the disk is less than I of the hoop.

2007 Free Response

Question 2

In March 1999 the Mars Global Surveyor (GS) entered its final orbit about Mars, sending data back to Earth. Assume a circular orbit with a period of 1.18×10^2 minutes $= 7.08 \times 10^3$ s and orbital speed of 3.40×10^3 m/s. The mass of the GS is 930 kg and the radius of Mars is 3.43×10^6 m.

(e) In fact, the orbit the GS entered was slightly elliptical with its closest approach to Mars at 3.71×10^5 m above the surface and its furthest distance at 4.36×10^5 m above the surface. If the speed of the GS at closest approach is 3.40×10^3 m/s, calculate the speed at the furthest point of the orbit.

For a correct expression of conservation of angular momentum

For a correct expression of conservation of angular momentum

$m_s v_1 r_1 = m_s v_2 r_2$ or equivalent such as $I_1 \omega_1 = I_2 \omega_2$ or $v_1 r_1 = v_2 r_2$

$v_2 = v_1 \dfrac{r_1}{r_2} = v_1 \dfrac{R_C \boxed{+ R_M}}{R_F \boxed{+ R_M}}$, where R_C and R_F are the distances of closest and farthest approaches, respectively, and R_M is the radius of Mars

For explicit substitution of radii (not altitudes) into the equation <u>or</u> for the correct numerical answer

$$v_2 = \left(3.40 \times 10^3 \text{ m/s}\right) \frac{3.71 \times 10^5 \text{ m} \boxed{+ 34.3 \times 10^5 \text{ m}}}{4.36 \times 10^5 \text{ m} \boxed{+ 34.3 \times 10^5 \text{ m}}}$$

$v_2 = 3.34 \times 10^3$ m/s

Alternatively, if the longer approach using conservation of energy was taken, 1 point was awarded for a correct statement of conservation of energy if explicitly written as $\frac{1}{2} m_s v_1^2 - \frac{G m_s M_M}{r_1} = \frac{1}{2} m_s v_2^2 - \frac{G m_s M_M}{r_2}$*, and 1 point was awarded for the explicit substitution of radii (not altitudes)* <u>*or*</u> *for a correct numerical answer.*

Free Response 2008-2010

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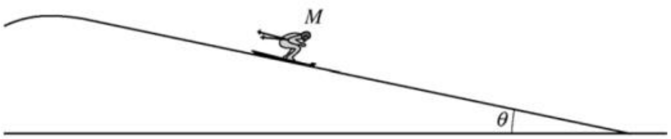
2008 Free Response

Question 1



Mech. 1.

A skier of mass M is skiing down a frictionless hill that makes an angle θ with the horizontal, as shown in the diagram. The skier starts from rest at time $t = 0$ and is subject to a velocity-dependent drag force due to air resistance of the form $F = -bv$, where v is the velocity of the skier and b is a positive constant. Express all algebraic answers in terms of M, b, θ, and fundamental constants.

(d) Solve the differential equation in part (b) to determine the velocity of the skier as a function of time, showing all your steps.

(d) 3 points

For taking the differential equation from part (b) and correctly separating the variables in preparation for integration (definite or indefinite integral)

$$M\frac{dv}{dt} = Mg\sin\theta - bv$$

$$\boxed{\frac{dv}{Mg\sin\theta - bv} = \frac{dt}{M}}$$

For correct integration of both sides of equation

For example, using a method involving an indefinite integral

Letting $u = Mg\sin\theta - bv$, so $du = -b\,dv$

$$-\frac{1}{b}\frac{du}{u} = \frac{dt}{M}$$

$$\int\frac{du}{u} = -\frac{b}{M}\int dt$$

$$\ln u = -\frac{b}{M}t + \ln C$$

$$u = Ce^{-bt/M}$$

$$\boxed{Mg\sin\theta - bv = Ce^{-bt/M}}$$

Using $v = 0$ at $t = 0$

$$Mg\sin\theta = C$$

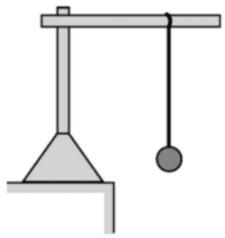
$$Mg\sin\theta - bv = Mg\sin\theta\, e^{-bt/M}$$

$$-bv = Mg\sin\theta\, e^{-bt/M} - Mg\sin\theta$$

For a correct final expression for $v(t)$

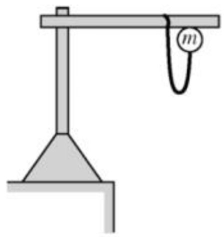
$$v = (Mg\sin\theta/b)\left(1 - e^{-bt/M}\right)$$

Question 3



Mech. 3.

In an experiment to determine the spring constant of an elastic cord of length 0.60 m, a student hangs the cord from a rod as represented above and then attaches a variety of weights to the cord. For each weight, the student allows the weight to hang in equilibrium and then measures the entire length of the cord. The data are recorded in the table below:



The student now attaches an object of unknown mass m to the cord and holds the object adjacent to the point at which the top of the cord is tied to the rod, as represented above. When the object is released from rest, it falls 1.5 m before stopping and turning around. Assume that air resistance is negligible.

iii. Calculate the maximum speed of the object.

(iii) 2 points

For a correct energy expression

$$\boxed{mgy_{\upsilon\max} = \frac{1}{2}kx^2 + \frac{1}{2}m\upsilon_{\max}{}^2}$$

$$\frac{1}{2}m\upsilon_{\max}{}^2 = mgy_{\upsilon\max} - \frac{1}{2}kx^2$$

$$\upsilon_{\max}{}^2 = 2gy_{\upsilon\max} - \frac{k}{m}x^2$$

For correct substitution of values previously obtained (especially those from part (d)(i))

$$\upsilon_{\max}{}^2 = 2(9.8\text{ m/s}^2)(0.87\text{ m}) - \frac{(25\text{ N/m})}{(0.69\text{ kg})}(0.27\text{ m})^2$$

$$\upsilon_{\max}{}^2 = 14.4(\text{m/s})^2$$

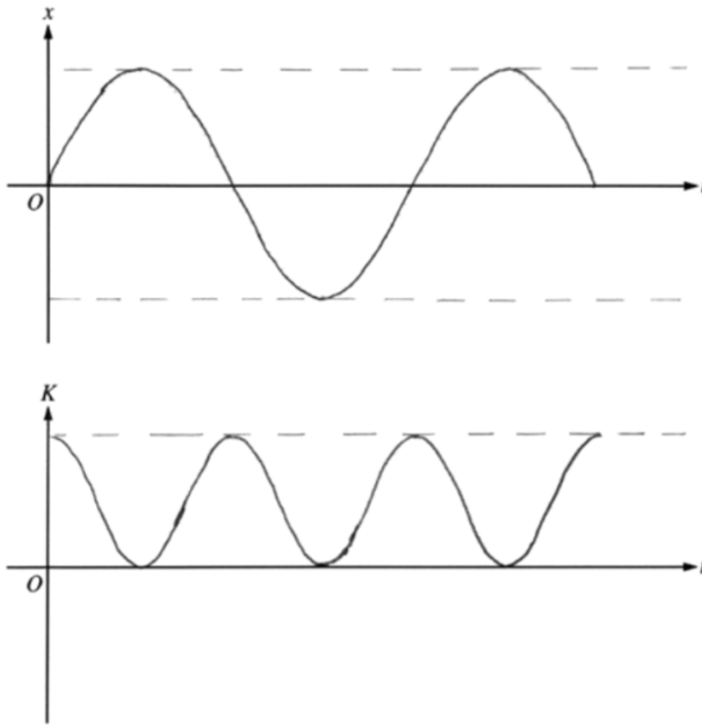
$$\upsilon_{\max} = 3.8\text{ m/s}$$

2009 Free Response

Question 1

Mech. 1.

A 3.0 kg object is moving along the x-axis in a region where its potential energy as a function of x is given as $U(x) = 4.0x^2$, where U is in joules and x is in meters. When the object passes the point $x = -0.50$ m, its velocity is +2.0 m/s. All forces acting on the object are conservative.

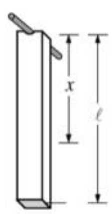


For a minimum of one complete cycle of a sine curve starting at the origin on the x versus t graph — 1 point

For a minimum of one complete cycle of a cosine squared curve starting at the maximum value on the K versus t graph — 1 point

For maxima and minima of the x graph matching the zeroes of the K graph — 1 point

Question 2



Mech. 2.

You are given a long, thin, rectangular bar of known mass M and length ℓ with a pivot attached to one end. The bar has a nonuniform mass density, and the center of mass is located a known distance x from the end with the pivot. You are to determine the rotational inertia I_b of the bar about the pivot by suspending the bar from the pivot, as shown above, and allowing it to swing. Express all algebraic answers in terms of I_b, the given quantities, and fundamental constants.

(a)

i. By applying the appropriate equation of motion to the bar, write the differential equation for the angle θ the bar makes with the vertical.

ii. By applying the small-angle approximation to your differential equation, calculate the period of the bar's motion.

(b) Describe the experimental procedure you would use to make the additional measurements needed to determine I_b. Include how you would use your measurements to obtain I_b and how you would minimize experimental error.

(c) Now suppose that you were not given the location of the center of mass of the bar. Describe an experimental procedure that you could use to determine it, including the equipment that you would need.

(a)

(i) 4 points

For the rotational form of Newton's second law
$\tau = I\alpha$

For a correct expression of the magnitude of torque
For correctly labeling the torque as negative
$-Mgx\sin\theta = I_b\alpha$

(a)

(i) 4 points

For the rotational form of Newton's second law
$\tau = I\alpha$
For a correct expression of the magnitude of torque
For correctly labeling the torque as negative
$-Mgx\sin\theta = I_b\alpha$
For expressing α as the second time derivative of θ

$$-Mgx\sin\theta = I_b\left(\frac{d^2\theta}{dt^2}\right)$$

(ii) 4 points

For the appropriate small angle approximation
For small angles, $\sin\theta \approx \theta$

$$-Mgx\theta = I_b\left(\frac{d^2\theta}{dt^2}\right)$$

$$\left(\frac{d^2\theta}{dt^2}\right) + \left(\frac{Mgx}{I_b}\right)\theta = 0$$

For recognizing that the coefficient of θ is ω^2

$$\omega^2 = \frac{Mgx}{I_b}$$

For the relationship between T and ω (this point was awarded for the equation alone or with relevant work, but NOT as part of multiple random equations)

$$\boxed{T = \frac{2\pi}{\omega}}$$

For the final expression for T (this point was awarded for the final correct answer with no supporting work)

$$T = 2\pi\sqrt{\frac{I_b}{Mgx}}$$

(b) 5 points

For an experimental procedure that includes:

A valid approach

How the variables will be measured or calculated, including equipment to be used

How these variables will be used to determine I_B

How to minimize error

Example 1: Displace the bar by a small angle and release it to oscillate. To reduce errors, time 10 complete oscillations with a stopwatch. Calculate the average value of the time for 10 oscillations and then divide by 10 to determine the period T. Calculate I_B from $T = 2\pi\sqrt{I_b/Mgx}$, using known values of M and x.

Example 2: Locate a photogate at the bottom of the bar's swing; set it to measure the amount of time the photogate is blocked. While the bar is hanging from its pivot point, displace the bar to a horizontal position and measure the height of the center of mass above the position of the photogate with a meter stick. Allow the bar to swing through the photogate and obtain the time the gate is blocked. To reduce errors, repeat this procedure five times and obtain an average time. Measure the width of the bar and use this and the time to determine the speed of the bar at the bottom of the swing, $\upsilon = \text{width}/\text{time}$. Calculate the angular speed of the bar from $\omega = \upsilon/\ell$. Apply conservation of energy to the bar: $Mgh = I_B\omega^2/2$. Calculate I_B from $I_B = 2Mgh/\omega^2$, using known values of M, measured value of h, and calculated value of ω.

(c) 2 points

For a valid procedure to locate the center of mass

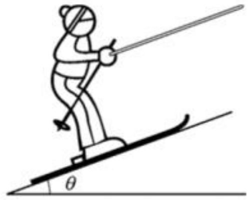
For specifying the equipment to be used

Example 1: Place the bar on top of a fulcrum, e.g., the top of a prism. Adjust the position of the bar until it is balanced horizontally. The point at which this occurs is the center of mass.

Example 2: Place the bar near the edge of a desk or table. Slowly push the bar so it hangs off the table until it is just ready to tip. The point at which this occurs is the center of mass.

2010 Free Response

Question 3



Mech. 3.

A skier of mass m will be pulled up a hill by a rope, as shown above. The magnitude of the acceleration of the skier as a function of time t can be modeled by the equations

$$
\begin{aligned}
a &= a_{\max} \sin \frac{\pi t}{T} \qquad (0 < t < T) \\
&= 0 \qquad\qquad\qquad (t \geq T),
\end{aligned}
$$

where $a_{\max}$ and T are constants. The hill is inclined at an angle θ above the horizontal, and friction between the skis and the snow is negligible. Express your answers in terms of given quantities and fundamental constants.

(a) Derive an expression for the velocity of the skier as a function of time during the acceleration. Assume the skier starts from rest.

(b) Derive an expression for the work done by the net force on the skier from rest until terminal speed is reached.

(c) Determine the magnitude of the force exerted by the rope on the skier at terminal speed.

(d) Derive an expression for the total impulse imparted to the skier during the acceleration.

(e) Suppose that the magnitude of the acceleration is instead modeled as $a = a_{\max} e^{-\pi t/2T}$ for all $t > 0$, where $a_{\max}$ and T are the same as in the original model. On the axes below, sketch the graphs of the force exerted by the rope on the skier for the two models, from $t = 0$ to a time $t > T$. Label the original model F_1 and the new model F_2.

(a) 4 points

For a correct relationship between velocity and acceleration

$$v = \int a(t)\,dt \quad \textbf{OR} \quad v = \int_0^t a(t)\,dt \quad \textbf{OR} \quad \frac{dv}{dt} = a$$

For a correct substitution of the expression for acceleration into the integral relationship

$$v = \int \left(a_{\max} \sin \frac{\pi t}{T}\right) dt \quad \textbf{OR} \quad v = \int_0^t \left(a_{\max} \sin \frac{\pi t}{T}\right) dt \qquad (0 < t < T)$$

For a correct evaluation of the integral, with an integration constant or correct limits

$$v = -\frac{a_{\max} T}{\pi} \cos \frac{\pi t}{T} + C \quad \textbf{OR} \quad v = -\frac{a_{\max} T}{\pi} \cos \frac{\pi t}{T}\Big|_0^t \qquad (0 < t < T)$$

For a correct determination of the integration constant or evaluation between the limits

$$\boxed{v(0) = -\frac{a_{\max} T}{\pi} + C = 0 \;\Rightarrow\; C = \frac{a_{\max} T}{\pi}} \quad \textbf{OR} \quad v = -\frac{a_{\max} T}{\pi}\left(\cos \frac{\pi t}{T} - 1\right) \qquad (0 < t < T)$$

$$v = \frac{a_{\max} T}{\pi}\left(1 - \cos \frac{\pi t}{T}\right) \qquad (0 < t < T)$$

(b) 2 points

For indicating that the work done by the net force is equal to the change in kinetic energy

$$\boxed{W = \frac{1}{2} m \left(v_f^2 - v_i^2\right)}$$

For a correct substitution of velocity from (a) into the work-energy expression

$$v_f = v_T = \frac{a_{\max} T}{\pi}(1 - \cos \pi) = \frac{2 a_{\max} T}{\pi}$$

$$v_i = v_0 = \frac{a_{\max} T}{\pi}(1 - \cos 0) = 0$$

$$W = \frac{1}{2} m \left(\frac{2 a_{\max} T}{\pi}\right)^2$$

$$W = \frac{2 m a_{\max}^2 T^2}{\pi^2}$$

(d) 2 points

$\boxed{J = \int F\,dt}$

For a correct substitution of force into the impulse-time relationship

$$J = ma_{\max} \int_0^T \sin\frac{\pi t}{T}\,dt$$

$$J = \frac{ma_{\max} T}{\pi}\left(-\cos\frac{\pi t}{T}\right)\Bigg|_0^T$$

For evaluation at the limits of integration

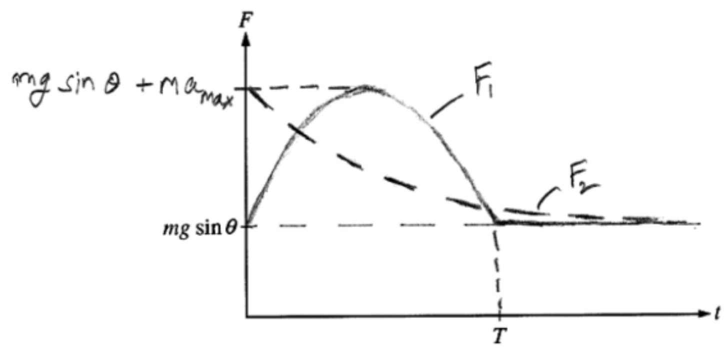
$$J = \frac{ma_{\max} T}{\pi}\left[-\cos\pi + \cos 0\right]$$

$$J = \frac{2ma_{\max} T}{\pi}$$

(e) 6 points

$$F_1 = mg\sin\theta + ma_{\max}\sin\left(\frac{\pi t}{T}\right) \quad (0 < t < T)$$

$$F_2 = mg\sin\theta + ma_{\max}\, e^{-\pi t/2T}$$



For a graph labeled F_1:

for starting at $mg\sin\theta$	1 point
for half a sine wave with a maximum at $\sim T/2$	1 point
for returning to original starting point at $t = T$	1 point
for a horizontal line at the original starting point for $t > T$	1 point

For a graph labeled F_2:

for starting on the vertical axis at a point above the starting point of F_1 (if there is no F_1 graph, this point was awarded if the F_2 graph starts above $mg\sin\theta$)	1 point
for an exponential decay graph	1 point

Free Response 2011-2013

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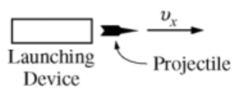
2011 Free Response

Question 1



Mech. 1.

A projectile is fired horizontally from a launching device, exiting with a speed v_x. While the projectile is in the launching device, the impulse imparted to it is J_p, and the average force on it is F_{avg}. Assume the force becomes zero just as the projectile reaches the end of the launching device. Express your answers to parts (a) and (b) in terms of v_x, J_p, F_{avg}, and fundamental constants, as appropriate.

(b) Determine an expression for the mass of the projectile.

For the correct relationship between impulse and the change in momentum

$$\mathbf{J} = \Delta\mathbf{p} = m\Delta\mathbf{v}$$

$$J_p = m(v_x - 0) = mv_x$$

For the correct answer

$$m = J_p / v_x$$

The projectile is fired horizontally into a block of wood that is clamped to a tabletop so that it cannot move. The projectile travels a distance d into the block before it stops. Express all algebraic answers to the following in terms of d and the given quantities previously indicated, as appropriate.

(c) Derive an expression for the work done in stopping the projectile.

(d) Derive an expression for the average force F_b exerted on the projectile as it comes to rest in the block.

(c) 3 points

For using the work-energy theorem

$$W = \Delta K$$

$$\boxed{W = 0 - \frac{1}{2} m v_x^2}$$

For substituting the expression for m from part (b)

$$W = -\frac{1}{2}\boxed{\frac{J_p}{v_x}} v_x^2$$

$$W = -\frac{1}{2} J_p v_x$$

For an indication that the work done is negative

Alternate Solution

Using kinematics and Newton's second law to determine the average net force

$$v_f^2 - v_i^2 = 2a_{avg} d$$

$$-v_x^2 = 2a_{avg} d$$

$$a_{avg} = -\frac{v_x^2}{2d}$$

$$\mathbf{F}_{avg} = m\mathbf{a}_{avg}$$

$$F_{net} = m\left(-\frac{v_x^2}{2d}\right)$$

For substituting this expression for the force into the equation for work

$$W = \int \mathbf{F} \cdot d\mathbf{r} = F_{avg} d = m\left(-\frac{v_x^2}{2d}\right) d$$

$$W = -m\frac{v_x^2}{2}$$

For substituting the expression for m from part (b)

$$W = -\frac{J_p}{v_x}\frac{v_x^2}{2}$$

$$W = -\frac{1}{2} J_p v_x$$

For an indication that the work done is negative

Now a new projectile and block are used, identical to the first ones, but the block is <u>not</u> clamped to the table. The projectile is again fired into the block of wood and travels a new distance d_n into the block while the block slides across the table a short distance D. Assume the following: the projectile enters the block with speed v_x, the average force F_b between the projectile and the block has the same value as determined in part (d), the average force of friction between the table and the block is f_T, and the collision is instantaneous so the frictional force is negligible during the collision.

(f) 2 points

For a correct application of conservation of momentum to the block-projectile collision

$$mv_x = (M + m)V$$

$$V = \frac{m}{(M + m)} v_x$$

The kinetic energy of the block/projectile system immediately after the collision is equal to the work done by friction in stopping it.

$$\frac{1}{2}(M + m)V^2 = f_T D$$

For substituting for V

$$\frac{1}{2}(M + m)\left(\frac{m}{(M + m)} v_x\right)^2 = f_T D$$

$$\frac{1}{2}\frac{m^2 v_x^2}{(M + m)} = f_T D$$

$$\frac{m}{M + m}\left(\frac{1}{2} m v_x^2\right) = f_T D$$

From part (c) the kinetic energy factor in the equation above is equal to the total work done. From part (d) that work is equal to $F_b d$.

$$\boxed{\frac{m}{M + m} F_b d = f_T D}$$

Using the expression $F_b d_n = F_b d - f_T D$ from part (e) to substitute for $f_T D$

$$\frac{m}{M + m} F_b d = F_b d - F_b d_n$$

$$\frac{m}{M + m} d = d - d_n$$

$$d_n = d\left(1 - \frac{m}{M + m}\right)$$

2012 Free Response

Question 2

You are to perform an experiment investigating the conservation of mechanical energy involving a transformation from initial gravitational potential energy to translational kinetic energy.

(a) You are given the equipment listed below, all the supports required to hold the equipment, and a lab table. On the list below, indicate each piece of equipment you would use by checking the line next to each item.

____ Track	____ Meterstick	____ Set of objects of different masses
____ Cart	____ Electronic balance	____ Lightweight low-friction pulley
____ String	____ Stopwatch	

(b) Outline a procedure for performing the experiment. Include a diagram of your experimental setup. Label the equipment in your diagram. Also include a description of the measurements you would make and a symbol for each measurement.

(c) Give a detailed account of the calculations of gravitational potential energy and translational kinetic energy both before and after the transformation, in terms of the quantities measured in part (b).

(d) After your first trial, your calculations show that the energy increased during the experiment. Assuming you made no mathematical errors, give a reasonable explanation for this result.

(e) On all other trials, your calculations show that the energy decreased during the experiment. Assuming you made no mathematical errors, give a reasonable explanation for the fact that the average energy you determined decreased. Include references to conservative and nonconservative forces, as appropriate.

(a) 1 point

For choosing the meterstick and stopwatch, regardless of what else is checked — 1 point

(b) 4 points

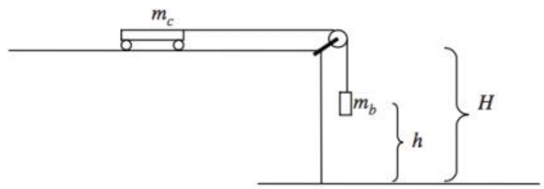
For a procedure that indicates the height needed to calculate gravitational potential energy — 1 point

For a procedure that indicates distance and time measurements to calculate velocity — 1 point

For a diagram and a clear indication of the height measurement — 1 point

For a diagram and a clear indication of the distance measurement — 1 point

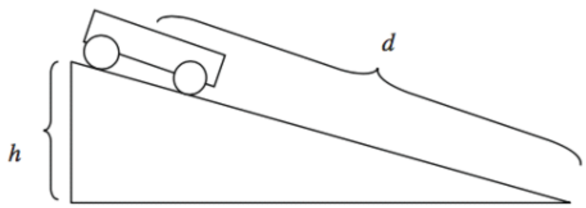
Example #1



- Use the electronic balance to determine the mass m_c of the cart and the mass m_b of one object.
- Attach the object to the cart using the string.
- Place the cart on the track and hang the object so that the string passes through the pulley.
- Allow the object to fall a distance h from its initial position to the floor, using the meterstick to measure the distance fallen.
- Use the stopwatch to measure the time t it takes the object to fall the distance h.
- Measure the height H of the table.

(b) continued

Example #2



- Use the electronic balance to determine the mass m of the cart.
- Set the track at an incline, and measure the height h of the incline.
- Place the cart at the top of the incline, and release from rest.
- Using the stopwatch, measure the time t it takes for the cart to move down the incline.
- Measure the distance d that the cart moves down the incline.

(c) 6 points

For a clear indication of the initial potential energy of the system	1 point
For a clear indication of the final potential energy of the system	1 point
For a clear indication of the initial kinetic energy of the system	1 point
For a clear indication of the final kinetic energy of the system	1 point
For a correct calculation of the instantaneous velocity of the system	2 points

Example #1

Initial gravitational potential energy: $U_{g0} = m_c gH + m_b gh$

Final gravitational potential energy: $U_{gf} = m_c gH$

Initial kinetic energy: $K_0 = 0$

Final kinetic energy: $K_f = \frac{1}{2}(m_c + m_b)v_f^2$

Acceleration is constant, so $d = \frac{1}{2}(v_0 + v_f)t$, where d is the distance along the track.

$v_f = \frac{2h}{t}$

(c) continued

Example #2

Initial gravitational potential energy: $U_{g0} = mgh$

Final gravitational potential energy $U_{gf} = 0$

Initial kinetic energy $K_0 = 0$

Final kinetic energy $K_f = \frac{1}{2}mv_f^2$

Acceleration is constant, so $d = \frac{1}{2}(v_0 + v_f)t$.

$v_f = \frac{2d}{t}$

(d) 2 points

For identifying a reasonable cause for the increase in energy

For a reasonable explanation related to the cause identified

Example

An unintentional push was applied to the cart, thus increasing the initial energy.

(e) 2 points

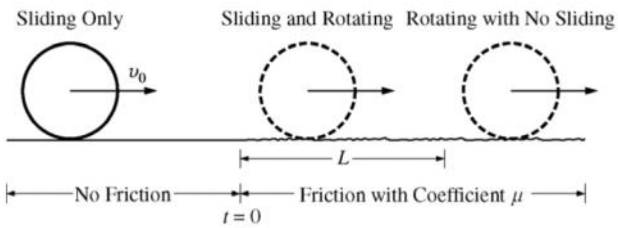
For identifying a reasonable cause for the decrease in energy related to the nonconservative forces acting on the system

For a reasonable explanation related to the cause identified

Example

Friction acting on the object decreases the speed, thereby decreasing the energy.

Question 3



Mech. 3.

A ring of mass M, radius R, and rotational inertia MR^2 is initially sliding on a frictionless surface at constant velocity v_0 to the right, as shown above. At time $t = 0$ it encounters a surface with coefficient of friction μ and begins sliding and rotating. After traveling a distance L, the ring begins rolling without sliding. Express all answers to the following in terms of M, R, v_0, μ, and fundamental constants, as appropriate.

(c) Derive an expression for the time it takes the ring to travel the distance L.

(d) Derive an expression for the magnitude of the velocity of the ring immediately after it has traveled the distance L.

(e) Derive an expression for the distance L.

(c) 2 points

For indicating that the linear speed is equal to $R\omega$ when the slipping stops

$v = R\omega$

$v_0 - \mu g t = R\left(\frac{\mu g t}{R}\right)$

For the correct answer

$t = \frac{v_0}{2\mu g}$

(d) 1 point

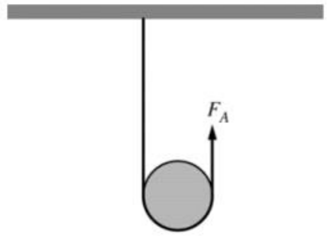
For substituting the time found in part (c) into a correct kinematics equation

$v = v_0 - \mu g\left(\frac{v_0}{2\mu g}\right)$

$v = v_0/2$

2013 Free Response

Question 3



Note: Figure not drawn to scale.

Mech 3.

A disk of mass $M = 2.0$ kg and radius $R = 0.10$ m is supported by a rope of negligible mass, as shown above. The rope is attached to the ceiling at one end and passes under the disk. The other end of the rope is pulled upward with a force F_A. The rotational inertia of the disk around its center is $MR^2/2$.

(a) Calculate the magnitude of the force F_A necessary to hold the disk at rest.

At time $t = 0$, the force F_A is increased to 12 N, causing the disk to accelerate upward. The rope does not slip on the disk as the disk rotates.

(b) Calculate the linear acceleration of the disk.

(c) Calculate the angular speed of the disk at $t = 3.0$ s.

(d) Calculate the increase in total mechanical energy of the disk from $t = 0$ to $t = 3.0$ s.

(e) The disk is replaced by a hoop of the same mass and radius. Indicate whether the linear acceleration of the hoop is greater than, less than, or the same as the linear acceleration of the disk.

__ Greater than   __ Less than   __ The same as

Justify your answer.

(a) 2 points

For a correct expression indicating that $F_{net} = 0$ — 1 point

$2F_A - Mg = 0$

$F_A = Mg/2$

$F_A = (2.0\text{ kg})(9.8\text{ m/s}^2)/2$

For a correct answer — 1 point

$F_A = 9.8\text{ N}$ (or 10 N using $g = 10\text{ m/s}^2$)

(b) 5 points

For a correct expression of Newton's second law for translational motion — 1point

$F_{net} = Ma$

$\boxed{F_A + T - Mg = Ma \text{ (equation 1)}}$

For using a correct expression of torque in Newton's second law for rotational motion — 1 point

$\tau = I\alpha$

$F_AR - TR = (MR^2/2)\alpha$ or $F_AR - TR = I\alpha$

For substituting for the angular acceleration in terms of the linear acceleration $(\alpha = a/R)$ — 1 point

$F_AR - TR = (MR^2/2)(a/R)$

$\boxed{F_A - T = Ma/2 \text{ (equation 2)}}$

For combining equations 1 and 2 to solve for the linear acceleration — 1 point

Add the two equations

$2F_A - Mg = (3/2)Ma$

$a = (2/3)((2F_A/M) - g)$

$a = (2/3)((2(12\text{ N})/2.0\text{ kg}) - 9.8\text{ m/s}^2)$

For a correct answer, with units — 1 point

$a = 1.47\text{ m/s}^2$ (1.33 m/s^2 using $g = 10\text{ m/s}^2$)

(c) 2 points

For using the relationship between linear and angular acceleration in the equation for angular speed

$\omega = \omega_0 + \alpha t$ and $\alpha = a/R$

$\omega_0 = 0$, so $\omega = at/R$

$\omega = \left(1.47\ \text{m/s}^2\right)(3.0\ \text{s})/(0.10\ \text{m})$

For an answer with units, consistent with previous work

$\omega = 44\ \text{rad/s}$ ($40\ \text{rad/s}$ using $g = 10\ \text{m/s}^2$)

(d) 4 points

Express the change in mechanical energy as the sum of the change in potential energy and the change in kinetic energy

$\Delta E = \Delta U_g + \Delta K$

For a correct expression for the change in kinetic energy including both translational and rotational kinetic energy — 1 point

$\boxed{\Delta K = \frac{1}{2}M\left(v^2 - v_0^2\right) + \frac{1}{2}I\left(\omega^2 - \omega_0^2\right)}$

For a correct expression for the change in potential energy, including a correct expression of the height h in terms of the time — 1 point

$\boxed{\Delta U_g = Mgh = Mg\left(\frac{1}{2}at^2\right)}$

v_0 and ω_0 are zero, so $\Delta E = Mg\left(\frac{1}{2}at^2\right) + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

For simplifying the expression using the relationship between linear velocity and angular velocity — 1 point

$\Delta E = Mg\left(\frac{1}{2}at^2\right) + \frac{1}{2}M(R\omega)^2 + \frac{1}{2}\left(MR^2/2\right)\omega^2$

$\Delta E = \frac{1}{2}Mgat^2 + \frac{3}{4}MR\omega^2$

For correctly substituting given values and answers from previous parts into a correct expression — 1 point

$\Delta E = \frac{1}{2}(2.0\ \text{kg})\left(9.8\ \text{m/s}^2\right)\left(1.47\ \text{m/s}^2\right)(3.0\ \text{s})^2 + \frac{3}{4}(2.0\ \text{kg})(0.10\ \text{m})^2(44\ \text{rad/s})^2$

$\Delta E = 159\ \text{J}$ ($144\ \text{J}$ using $g = 10\ \text{m/s}^2$)

(e) 2 points

For selecting "Less than" — 1 point

For a correct justification — 1 point

Example

The rotational inertia of a hoop is greater than that of a solid disk of the same mass and radius, therefore the acceleration of the hoop would be less.

Free Response 2014-2016

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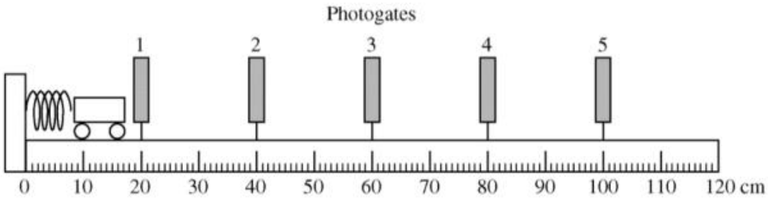
2014 Free Response

Question 1



Mech. 1.

In an experiment, a student wishes to use a spring to accelerate a cart along a horizontal, level track. The spring is attached to the left end of the track, as shown in the figure above, and produces a nonlinear restoring force of magnitude $F_s = As^2 + Bs$, where s is the distance the spring is compressed, in meters. A measuring tape, marked in centimeters, is attached to the side of the track. The student places five photogates on the track at the locations shown.

(d)

i. Compare the speed of the cart measured by photogate 1 to the predicted value of the speed of the cart just after it loses contact with the spring. List a physical source of error that could account for the difference.

ii. From the measured speed values of the cart as it rolls down the track, give a physical explanation for any trend you observe.

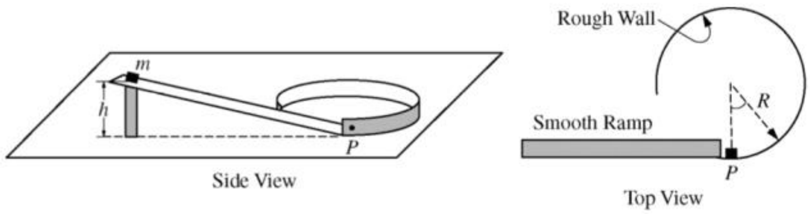
(d) i. 2 points

For stating that the measured initial speed of the cart is greater than the predicted value
For correctly identifying a source of error regarding the initial speed of the cart
Examples:
The student compressed the spring more than was determined. This would lead to more potential energy in the spring and greater kinetic energy for the cart. The cart would therefore move faster than predicted.
The table is not level, sloping downward would result in a greater measured speed.
The constants A and B for the spring are not accurate. The true values are larger than what is given. This would lead to smaller predicted potential energy of the spring and a smaller predicted value for the kinetic energy of the cart. Therefore, the cart would move faster than predicted

ii. 2 points

For correctly identifying the trend
For a correct physical explanation for the cart slowing down
Examples:
Friction in the axles and air resistance against the cart are slowing it down.
The track is not perfectly level and the cart is going uphill. This is slowing down the cart.

Question 2



Mech. 2.

A small block of mass m starts from rest at the top of a frictionless ramp, which is at a height h above a horizontal tabletop, as shown in the side view above. The block slides down the smooth ramp and reaches point P with a speed v_0. After the block reaches point P at the bottom of the ramp, it slides on the tabletop guided by a circular vertical wall with radius R, as shown in the top view. The tabletop has negligible friction, and the coefficient of kinetic friction between the block and the circular wall is μ.

(e) Derive an expression for $v(t)$, the speed of the block as a function of time t after passing point P on the track.

(e) 3 points

For substituting dv/dt for a into the answer from part (d), or substituting dv/dt for a and the friction force for F_{net} into Newton's second law

$$\frac{dv}{dt} = -\frac{\mu v^2}{R} \quad \text{or} \quad m\frac{dv}{dt} = -F_f$$

For including the negative sign

Substituting for F_f produces the same relationship as the first equation above.

For separation of variables and using correct limits

$$\frac{1}{v^2}dv = -\frac{\mu}{R}dt$$

$$\int_{v_0}^{v} \frac{1}{v^2}dv = \int_0^t -\frac{\mu}{R}dt$$

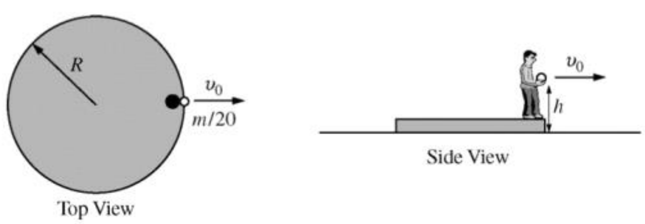
Integrate the equation to solve for v.

$$\left[-\frac{1}{v}\right]_{v_0}^{v} = \left[-\frac{\mu t}{R}\right]_0^t$$

$$\frac{1}{v} - \frac{1}{v_0} = \frac{\mu t}{R}$$

$$v = \frac{Rv_0}{R + \mu v_0 t} \quad \text{or} \quad \frac{v_0}{1 + \mu v_0 t/R}$$

Question 3



Mech. 3.

A large circular disk of mass m and radius R is initially stationary on a horizontal icy surface. A person of mass $m/2$ stands on the edge of the disk. Without slipping on the disk, the person throws a large stone of mass $m/20$ horizontally at initial speed v_0 from a height h above the ice in a radial direction, as shown in the figures above. The coefficient of friction between the disk and the ice is μ. All velocities are measured relative to the ground. The time it takes to throw the stone is negligible. Express all algebraic answers in terms of m, R, v_0, h, μ, and fundamental constants, as appropriate.

(b) Assuming that the disk is free to slide on the ice, derive an expression for the speed of the disk and person immediately after the stone is thrown.

(b) 3 points

For a statement of conservation of momentum or Newton's third law

$p_i = p_f$

For substituting the momentum of the stone into a correct expression for conservation of momentum

For substituting the momentum of the person-disk system into a correct expression for conservation of momentum

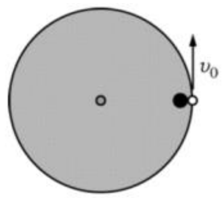
$0 = m_1 v_1 + m_2 v_2$

$0 = \left(\frac{m}{20}\right)(v_0) + \left(m + \frac{m}{2}\right)v$

$\frac{3}{2}mv = -\frac{1}{20}mv_0$

$v = -\frac{1}{30}v_0$

Note: Since the question asks for speed, the negative sign is not needed. There is no penalty for including it.



Top View

The person now stands on a similar disk of mass m and radius R that has a fixed pole through its center so that it can only rotate on the ice. The person throws the same stone horizontally in a tangential direction at initial speed v_0, as shown in the figure above. The rotational inertia of the disk is $mR^2/2$.

(d) Derive an expression for the angular speed ω of the disk immediately after the stone is thrown.

(d) 4 points

For a statement of conservation of total angular momentum

$L_i = L_f$

$L = mrv$ for linear motion

$L = I\omega$ for rotation

For substituting the angular momentum of the stone into a correct expression of conservation of angular momentum

For substituting the angular momentum of the person into a correct expression of conservation of angular momentum

For substituting the angular momentum of the disk into a correct expression of conservation of angular momentum

$\boxed{0 = m_s r_s v_s + I_D \omega_D + I_P \omega_P}$

$0 = \left(\frac{m}{20}Rv_0\right) - \left(\frac{mR^2}{2}\omega + \frac{m}{2}R^2\omega\right)$

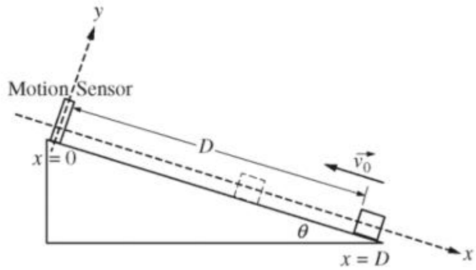
$\left(\frac{m}{20}Rv_0\right) = \left(\frac{mR^2}{2}\omega + \frac{m}{2}R^2\omega\right) = mR^2\omega$

$\omega = \frac{\frac{m}{20}Rv_0}{mR^2}$

$\omega = \frac{v_0}{20R}$

2015 Free Response

Question 1



Mech.1.

A block of mass m is projected up from the bottom of an inclined ramp with an initial velocity of magnitude v_0. The ramp has negligible friction and makes an angle θ with the horizontal. A motion sensor aimed down the ramp is mounted at the top of the incline so that the positive direction is down the ramp. The block starts a distance D from the motion sensor, as shown above. The block slides partway up the ramp, stops before reaching the sensor, and then slides back down.

(b) Derive an expression for the position x_{min} of the block when it is closest to the motion sensor. Express your answer in terms of m, D, v_0, θ, and physical constants, as appropriate.

(b) 2 points

Using an equation that can be solved for the closest position to the sensor

$$v_2^2 = v_1^2 + 2ad$$

For substitution into a correct kinematic equation consistent with part (a)

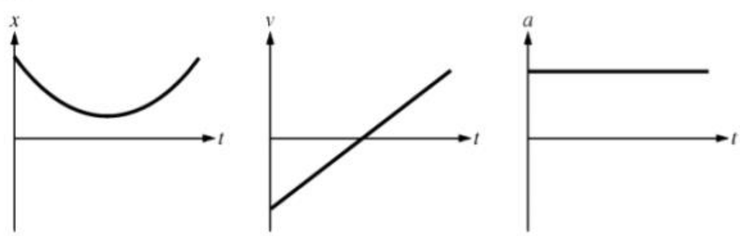
For setting v_2 to zero and using D for the initial position

$$0 = v_0^2 + 2(g\sin\theta)\boxed{(x - D)}$$

$$x = D - \frac{v_0^2}{2g\sin\theta}$$

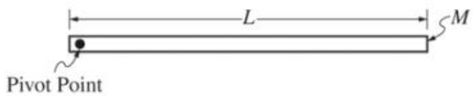
(c) On the axes provided below, sketch graphs of position x, velocity v, and acceleration a as functions of time t for the motion of the block while it goes up and back down the ramp. Explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expressions, as appropriate.

(c) 4 points



For a position graph that is a parabola that does not cross the t-axis and has a vertex that does not touch the t-axis	1 point
For a velocity graph that is a straight line and crosses the t-axis	1 point
For an acceleration graph that is a horiz	1 point
For a set of graphs that are consistent with each other	1 point

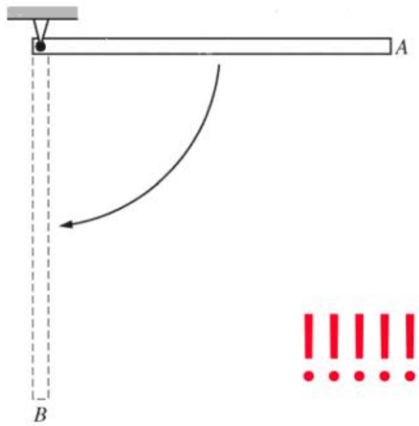
Question 3



Mech.3.

A uniform, thin rod of length L and mass M is allowed to pivot about its end, as shown in the figure above.

(a) Using integral calculus, derive the rotational inertia for the rod around its end to show that it is $ML^2/3$.



!!!!!!!

(a) 3 points

Writing an integral to derive the rotational inertia of the rod

$$I = \int r^2 dm$$

For a correct expression for dm

$$\lambda = M/L,\ M = \lambda L,\ dm = \lambda dr$$

For using the correct limits of integration or a correct constant of integration

$$I = \int_{r=0}^{r=L} \lambda r^2 dr$$

For correctly evaluating the integral above, leading to the answer $ML^2/3$

$$I = \left[\frac{\lambda r^3}{3}\right]_{r=0}^{r=L} = \frac{1}{3}\lambda\left(L^3 - 0\right) = \frac{1}{3}\left(\frac{M}{L}\right)\left(L^3\right) = \frac{1}{3}ML^2$$

(b) Derive an expression for the velocity of the free end of the rod at position B. Express your answer in terms of M, L, and physical constants, as appropriate.

(b) 4 points

For using any expression of conservation of energy

$$K_1 + U_{g1} = K_2 + U_{g2}$$

For a correct energy expression relating gravitational potential energy to rotational kinetic energy

$$mgh_1 = \boxed{\frac{1}{2}I\omega_2^2}$$

For correctly substituting $L/2$ for the change in height

$$Mg(L/2) = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2$$

For using $\boxed{v = r\omega}$ with $r = L$ to solve for the velocity of the end of the rod

$$\frac{MgL}{2} = \frac{1}{6}ML^2\left(\frac{v}{L}\right)^2$$

$$v = \sqrt{3gL}$$

ii. Describe two ways in which the effects of air resistance could be reduced.

For one example that directly decreases the effect of air resistance
For another example that directly decreases the effect of air resistance
Some examples include:

- Do the experiment in a vacuum
- Use shorter rod lengths
- Use more massive (or denser) rods
- Use a more aerodynamic shape for the rods

2016 Free Response

Question 2

Mech.2.

A block of mass $2M$ rests on a horizontal, frictionless table and is attached to a relaxed spring, as shown in the figure above. The spring is nonlinear and exerts a force $F(x) = -Bx^3$, where B is a positive constant and x is the displacement from equilibrium for the spring. A block of mass $3M$ and initial speed v_0 is moving to the left as shown.

(d) Derive an expression for the maximum distance D that the spring is compressed.

(d) 4 points

For a correct expression of the conservation of energy

$\Delta K_{system} + \Delta U_{system} = 0$

$K_0 = U_{final}$

For attempting to integrate the spring force equation

$$\frac{9}{10}Mv_0^2 = -\int_{x_1}^{x_2} -Bx^3 dx$$

$$\frac{9}{10}Mv_0^2 = \int_{x_1}^{x_2} Bx^3 dx$$

For using the correct limits of integration or an appropriate constant of integration

$$\frac{9}{10}Mv_0^2 = \int_0^D Bx^3 dx$$

$$\frac{9}{10}Mv_0^2 = \left[\frac{Bx^4}{4}\right]_0^D$$

For an answer consistent with the speed from (b) or the kinetic energy from part (c)

$$D = \sqrt[4]{\frac{18Mv_0^2}{5B}}$$

ii. Which of the following correctly describes the magnitude of the net force on each of the two blocks when the spring is at maximum compression?

____ The magnitude of the net force is greater on the block of mass $2M$.

____ The magnitude of the net force is greater on the block of mass $3M$.

____ The magnitude of the net force on each block has the same nonzero value.

____ The magnitude of the net force on each block is zero.

(e)

ii. 2 points

The magnitude of the net force is greater on the block of mass $3M$.
If the incorrect selection is made, no points are earned for the justification.
For an indication that both blocks will have the same acceleration
For a correct justification for why the net force is greater on the block of mass $3M$
Example:
Because the blocks stick together, both blocks must have the same acceleration.
Because the block of mass $3M$ has more mass, the net force on it must be greater than the net force on the block of mass $2M$.